

# A conditional limit theorem for random walks under extreme deviation

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## Abstract

This paper explores a conditional Gibbs theorem for a random walk induced by i.i.d.  $(X_1, \dots, X_n)$  conditioned on an extreme deviation of its sum ( $S_1^n = na_n$ ) or ( $S_1^n > na_n$ ) where  $a_n \rightarrow \infty$ . It is proved that when the summands have light tails with some additional regularity property, then the asymptotic conditional distribution of  $X_1$  can be approximated in variation norm by the tilted distribution at point  $a_n$ , extending therefore the classical LDP case.

## 1 Introduction

Let  $X_1^n := (X_1, \dots, X_n)$  denote  $n$  independent unbounded real valued random variables and  $S_1^n := X_1 + \dots + X_n$  denote their sum. The purpose of this paper is to explore the limit distribution of the generic variable  $X_1$  conditioned on extreme deviations (ED) pertaining to  $S_1^n$ . By extreme deviation we mean that  $S_1^n/n$  is supposed to take values which are going to infinity as  $n$  increases. Obviously such events are of infinitesimal probability. Our interest in this question stems from a first result which assesses that under appropriate conditions, when the sequence  $a_n$  is such that

$$\lim_{n \rightarrow \infty} a_n = \infty$$

then there exists a sequence  $\varepsilon_n$  which tends to 0 as  $n$  tends to infinity such that

$$\lim_{n \rightarrow \infty} P(\cap_{i=1}^n (X_i \in (a_n - \varepsilon_n, a_n + \varepsilon_n)) | S_1^n/n > a_n) = 1 \quad (1.1)$$

which is to say that when the empirical mean takes exceedingly large values, then all the summands share the same behaviour. This result obviously requires a number of

hypotheses, which we simply quote as “light tails” type. We refer to [3] for this result and the connection with earlier related works.

The above result is clearly to be put in relation with the so-called Gibbs conditional Principle which we recall briefly in its simplest form.

Consider the case when the sequence  $a_n = a$  is constant with value larger than the expectation of  $X_1$ . Hence we consider the behaviour of the summands when  $(S_1^n/n > a)$ , under a large deviation (LD) condition about the empirical mean. The asymptotic conditional distribution of  $X_1$  given  $(S_1^n/n > a)$  is the well known tilted distribution of  $P_X$  with parameter  $t$  associated to  $a$ . Let us introduce some notation to put this in light. The hypotheses to be stated now together with notation are kept throughout the entire paper.

It will be assumed that  $P_X$ , which is the distribution of  $X_1$ , has a density  $p$  with respect to the Lebesgue measure on  $\mathbb{R}$ . The fact that  $X_1$  has a light tail is captured in the hypothesis that  $X_1$  has a moment generating function

$$\Phi(t) := E \exp tX_1$$

which is finite in a non void neighborhood  $\mathcal{N}$  of 0. This fact is usually referred to as a Cramer type condition.

Defined on  $\mathcal{N}$  are the following functions. The functions

$$t \rightarrow m(t) := \frac{d}{dt} \log \Phi(t)$$

$$t \rightarrow s^2(t) := \frac{d}{dt} m(t)$$

$$t \rightarrow \mu_j(t) := \frac{d}{dt} s^2(t) \quad , \quad j = 3, 4$$

are the expectation and the three first centered moments of the r.v.  $\mathcal{X}_t$  with density

$$\pi_t(x) := \frac{\exp tx}{\Phi(t)} p(x)$$

which is defined on  $\mathbb{R}$  and which is the tilted density with parameter  $t$ . When  $\Phi$  is steep, meaning that

$$\lim_{t \rightarrow t^+} m(t) = \infty$$

where  $t^+ := \text{ess sup } \mathcal{N}$  then  $m$  parametrizes the convex hull of the support of  $P_X$ . We refer to Barndorff-Nielsen (1978) for those properties. As a consequence of this fact, for all  $a$  in the support of  $P_X$ , it will be convenient to define

$$\pi^a = \pi_t$$

where  $a$  is the unique solution of the equation  $m(t) = a$ .

We now come to some remark on the Gibbs conditional principle in the standard above setting. A phrasing of this principle is:

As  $n$  tends to infinity the conditional distribution of  $X_1$  given  $(S_1^n/n > a)$  is  $\Pi^a$ , the distribution with density  $\pi^a$ .

Indeed we prefer to state Gibbs principle in a form where the conditioning event is a point condition  $(S_1^n/n = a)$ . The conditional distribution of  $X_1$  given  $(S_1^n/n = a)$  is a well defined distribution and Gibbs conditional principle states that this conditional distribution converges to  $\Pi^a$  as  $n$  tends to infinity. In both settings, this convergence holds in total variation norm. We refer to [6] for the local form of the conditioning event; we will mostly be interested in the extension of this form in the present paper.

For all  $\alpha$  (depending on  $n$  or not) we will denote  $p_\alpha$  the density of the random vector  $X_1^k$  conditioned upon the local event  $(S_1^n = n\alpha)$ . The notation  $p_\alpha(X_1^k = x_1^k)$  is sometimes used to denote the value of the density  $p_\alpha$  at point  $x_1^k$ . The same notation is used when  $X_1, \dots, X_n$  are sampled under some  $\Pi^\alpha$ , namely  $\pi^\alpha(X_1^k = x_1^k)$ .

In [4] some extension of the above Gibbs principle has been obtained. When  $a_n = a > EX_1$  a second order term provides a sharpening of the conditioned Gibbs principle, stating that

$$\lim_{n \rightarrow \infty} \int |p_a(x) - g_a(x)| dx = 0 \quad (1.2)$$

where

$$g_a(x) := Cp(x)\mathbf{n}(a, s_n^2, x). \quad (1.3)$$

Hereabove  $\mathbf{n}(a, s_n, x)$  denotes the normal density function at point  $x$  with expectation  $a$ , with variance  $s_n^2$ , and  $s_n^2 := s^2(t)(n-1)$ . In the above display,  $C$  is a normalizing constant. Obviously developing in this display yields

$$g_a(x) = \pi^a(x)(1 + o(1))$$

which proves that (1.2) is a weak form of Gibbs principle, with some improvement due to the second order term.

The paper is organized as follows. Notation and hypotheses are stated in Section 2, along with some necessary facts from asymptotic analysis in the context of light tailed densities. Section 3 provides a local Gibbs conditional principle under EDP, namely producing the approximation of the conditional density of  $X_1, \dots, X_k$  conditionally on  $((1/n)(X_1 + \dots + X_n) = a_n)$  for sequences  $a_n$  which tend to infinity, and where  $k$  is fixed, independent on  $n$ . The approximation is local. This result is extended in Section 4 to typical paths under the conditional sampling scheme, which in turn provides the approximation in variation norm for the conditional distribution; in this extension,  $k$  is equal 1, although the result clearly also holds for fixed  $k > 1$ . The method used here follows closely

the approach by [4]. Discussion of the differences between the Gibbs principles in LDP and EDP are discussed. Section 5 states similar results in the case when the conditioning event is  $((1/n)(X_1 + \dots + X_n) > a_n)$ .

The main tools to be used come from asymptotic analysis and local limit theorems, developed from [7] and [1]; we also have borrowed a number of arguments from [9]. A number of technical lemmas have been postponed to the appendix.

## 2 Notation and hypotheses

In this paper, we consider the uniformly bounded density function  $p(x)$

$$p(x) = c \exp \left( - (g(x) - q(x)) \right) \quad x \in \mathbb{R}_+, \quad (2.1)$$

where  $c$  is some positive normalized constant. Define  $h(x) := g'(x)$ . We assume that for some And there exists some positive constant  $\vartheta$ , for large  $x$ , it holds

$$\sup_{|v-x| < \vartheta x} |q(v)| \leq \frac{1}{\sqrt{xh(x)}}. \quad (2.2)$$

The function  $g$  is positive and satisfies

$$\frac{g(x)}{x} \longrightarrow \infty, \quad x \rightarrow \infty. \quad (2.3)$$

Not all positive  $g$ 's satisfying (2.3) are adapted to our purpose. Regular functions  $g$  are defined as follows. We define firstly a subclass  $R_0$  of the family of *slowly varying* function. A function  $l$  belongs to  $R_0$  if it can be represented as

$$l(x) = \exp \left( \int_1^x \frac{\epsilon(u)}{u} du \right), \quad x \geq 1, \quad (2.4)$$

where  $\epsilon(x)$  is twice differentiable and  $\epsilon(x) \rightarrow 0$  as  $x \rightarrow \infty$ .

We follow the line of Juszczak and Nagaev [9] to describe the assumed regularity conditions of  $h$ .

**Class  $R_\beta$  :**  $h(x) \in R_\beta$ , if, with  $\beta > 0$  and  $x$  large enough,  $h(x)$  can be represented as

$$h(x) = x^\beta l(x),$$

where  $l(x) \in R_0$  and in (2.4)  $\epsilon(x)$  satisfies

$$\limsup_{x \rightarrow \infty} x |\epsilon'(x)| < \infty, \quad \limsup_{x \rightarrow \infty} x^2 |\epsilon''(x)| < \infty. \quad (2.5)$$

**Class  $R_\infty$  :** Further,  $l \in \widetilde{R}_0$ , if, in (2.4),  $l(x) \rightarrow \infty$  as  $x \rightarrow \infty$  and

$$\lim_{x \rightarrow \infty} \frac{x\epsilon'(x)}{\epsilon(x)} = 0, \quad \lim_{x \rightarrow \infty} \frac{x^2\epsilon''(x)}{\epsilon(x)} = 0, \quad (2.6)$$

and, for some  $\eta \in (0, 1/4)$

$$\liminf_{x \rightarrow \infty} x^\eta \epsilon(x) > 0. \quad (2.7)$$

We say that  $h \in R_\infty$  if  $h$  is increasing and strictly monotone and its inverse function  $\psi$  defined through

$$\psi(u) := h^\leftarrow(u) := \inf \{x : h(x) \geq u\} \quad (2.8)$$

belongs to  $\widetilde{R}_0$ .

Denote  $\mathfrak{R} := R_\beta \cup R_\infty$ . In fact,  $\mathfrak{R}$  covers one large class of functions, although,  $R_\beta$  and  $R_\infty$  are only subsets of *Regularly varying* and *Rapidly varying* functions, respectively.

**Remark 2.1.** *The rôle of (2.4) is to make  $h(x)$  smooth enough. Under (2.4) the third order derivative of  $h(x)$  exists, which is necessary in order to use a Laplace methode for the asymptotic evaluation of the moment generating function  $\Phi(t)$  as  $t \rightarrow \infty$ , where*

$$\Phi(t) = \int_0^\infty e^{tx} p(x) dx = c \int_0^\infty \exp \left( K(x, t) + q(x) \right) dx, \quad t \in (0, \infty)$$

in which

$$K(x, t) = tx - g(x).$$

If  $h \in \mathfrak{R}$ ,  $K(x, t)$  is concave with respect to  $x$  and takes its maximum at  $\hat{x} = h^\leftarrow(t)$ . The evaluation of  $\Phi(t)$  for large  $t$  follows from an expansion of  $K(x, t)$  in a neighborhood of  $\hat{x}$ ; this is Laplace's method. This expansion yields

$$K(x, t) = K(\hat{x}, t) - \frac{1}{2} h'(\hat{x}) (x - \hat{x})^2 - \frac{1}{6} h''(\hat{x}) (x - \hat{x})^3 + \epsilon(x, t),$$

where  $\epsilon(x, t)$  is some error term. Conditions (2.6) (2.7) and (2.5) guarantee that  $\epsilon(x, t)$  goes to 0 when  $t$  tends to  $\infty$  when  $x$  belongs to some neighborhood of  $\hat{x}$ .

**Example 2.1. Weibull Density.** Let  $p$  be a Weibull density with shape parameter  $k > 1$  and scale parameter 1, namely

$$\begin{aligned} p(x) &= kx^{k-1} \exp(-x^k), \quad x \geq 0 \\ &= k \exp \left( - (x^k - (k-1) \log x) \right). \end{aligned}$$

Take  $g(x) = x^k - (k-1) \log x$  and  $q(x) = 0$ . Then it holds

$$h(x) = kx^{k-1} - \frac{k-1}{x} = x^{k-1} \left( k - \frac{k-1}{x^k} \right).$$

Set  $l(x) = k - (k - 1)/x^k, x \geq 1$ , then (2.4) holds, namely,

$$l(x) = \exp\left(\int_1^x \frac{\epsilon(u)}{u} du\right), \quad x \geq 1,$$

with

$$\epsilon(x) = \frac{k(k-1)}{kx^k - (k-1)}.$$

The function  $\epsilon$  is twice differentiable and goes to 0 as  $x \rightarrow \infty$ . Additionally,  $\epsilon$  satisfies condition (2.5). Hence we have shown that  $h \in R_{k-1}$ .

**Example 2.2. A rapidly varying density.** Define  $p$  through

$$p(x) = c \exp(-e^{x-1}), \quad x \geq 0.$$

Then  $g(x) = h(x) = e^x$  and  $q(x) = 0$  for all non negative  $x$ . We show that  $h \in R_\infty$ . It holds  $\psi(x) = \log x + 1$ . Since  $h(x)$  is increasing and monotone, it remains to show that  $\psi(x) \in \widehat{R}_0$ . When  $x \geq 1$ ,  $\psi(x)$  admits the representation of (2.4) with  $\epsilon(x) = \log x + 1$ . Also conditions (2.6) and (2.7) are satisfied. Thus  $h \in R_\infty$ .

Throughout the paper we use the following notation. When a r.v.  $X$  has density  $p$  we write  $p(X = x)$  instead of  $p(x)$ . This notation is useful when changing measures. For example  $\pi^a(X = x)$  is the density at point  $x$  for the variable  $X$  generated under  $\pi^a$ , while  $p(X = x)$  states for  $X$  generated under  $p$ . This avoids constant changes of notation.

### 3 Conditional Density

We inherit of the definition of the tilted density  $\pi^a$  defined in Section 1, and of the corresponding definitions of the functions  $m$ ,  $s^2$  and  $\mu_3$ . Because of (2.1) and on the various conditions on  $g$  those functions are defined as  $t \rightarrow \infty$ . The following Theorem is basic for the proof of the remaining results.

**Theorem 3.1.** Let  $p(x)$  be defined as in (2.1) and  $h(x) \in \mathfrak{R}$ . Denote by

$$m(t) = \frac{d}{dt} \log \Phi(t), \quad s^2(t) = \frac{d}{dt} m(t), \quad \mu_3(t) = \frac{d^3}{dt^3} \log \Phi(t),$$

then with  $\psi$  defined as in (2.8) it holds as  $t \rightarrow \infty$

$$m(t) \sim \psi(t), \quad s^2(t) \sim \psi'(t), \quad \mu_3(t) \sim \frac{M_6 - 3}{2} \psi''(t),$$

where  $M_6$  is the sixth order moment of standard normal distribution.

The proof of this result relies on a series of Lemmas. Lemmas (7.2), (7.3), (7.4) and (7.5) are used in the proof. Lemma (7.1) is instrumental for Lemma (7.5). The proof of Theorem 3.1 and these Lemmas are postponed to Appendix.

**Corollary 3.1.** *Let  $p(x)$  be defined as in (2.1) and  $h(x) \in \mathfrak{R}$ . Then it holds as  $t \rightarrow \infty$*

$$\frac{\mu_3(t)}{s^3(t)} \rightarrow 0. \quad (3.1)$$

Proof: Its proof relies on Theorem 2.1 and is also put in Appendix.

## 4 Edgeworth expansion under extreme normalizing factors

With  $\pi^{a_n}$  defined through

$$\pi^{a_n}(x) = \frac{e^{tx}p(x)}{\Phi(t)},$$

and  $t$  determined by  $a_n = m(t)$ , define the normalized density of  $\pi^{a_n}$  by

$$\bar{\pi}^{a_n}(x) = s_n \pi^{a_n}(s_n x + a_n),$$

and denote the  $n$ -convolution of  $\bar{\pi}^{a_n}(x)$  by  $\bar{\pi}_n^{a_n}(x)$ . Denote by  $\rho_n$  the normalized density of  $n$ -convolution  $\bar{\pi}_n^{a_n}(x)$ ,

$$\rho_n(x) := \sqrt{n} \bar{\pi}_n^{a_n}(\sqrt{n}x).$$

The following result extends the local Edgeworth expansion of the distribution of normalized sums of i.i.d. r.v's to the present context, where the summands are generated under the density  $\bar{\pi}^{a_n}$ . Therefore the setting is that of a triangular array of row wise independent summands; the fact that  $a_n \rightarrow \infty$  makes the situation unusual. We mainly adapt Feller's proof (Chapter 16, Theorem 2 [7]).

**Theorem 4.1.** *With the above notation, uniformly upon  $x$  it holds*

$$\rho_n(x) = \phi(x) \left( 1 + \frac{\mu_3}{6\sqrt{n}s^3} (x^3 - 3x) \right) + o\left(\frac{1}{\sqrt{n}}\right).$$

where  $\phi(x)$  is standard normal density.

Proof: **Step 1:** In this step, we will express the following formula  $G(x)$  by its Fourier transform. Let

$$G(x) := \rho_n(x) - \phi(x) - \frac{\mu_3}{6\sqrt{n}s_n^3} (x^3 - 3x) \phi(x).$$

From

$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\tau x} e^{-\frac{1}{2}\tau^2} d\tau, \quad (4.1)$$

it follows that

$$\phi'''(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} (i\tau)^3 e^{-i\tau x} e^{-\frac{1}{2}\tau^2} d\tau. \quad (4.2)$$

On the other hand

$$\phi'''(x) = -(x^3 - 3x)\phi(x),$$

which, together with (4.2), gives

$$(x^3 - 3x)\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\tau)^3 e^{-i\tau x} e^{-\frac{1}{2}\tau^2} d\tau. \quad (4.3)$$

Let  $\varphi^{a_n}(\tau)$  be the characteristic function (c.f) of  $\bar{\pi}^{a_n}$ ; the c.f of  $\rho_n$  is  $(\varphi^{a_n}(\tau/\sqrt{n}))^n$ . Hence it holds by Fourier inversion theorem

$$\rho_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\tau x} (\varphi^{a_n}(\tau/\sqrt{n}))^n d\tau. \quad (4.4)$$

Using (4.1), (4.3) and (4.4), we have

$$G(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\tau x} \left( (\varphi^{a_n}(\tau/\sqrt{n}))^n - e^{-\frac{1}{2}\tau^2} - \frac{\mu_3}{6\sqrt{n}s^3} (i\tau)^3 e^{-\frac{1}{2}\tau^2} \right) d\tau.$$

Hence it holds

$$\begin{aligned} & \left| \rho_n(x) - \phi(x) - \frac{\mu_3}{6\sqrt{n}s^3} (x^3 - 3x)\phi(x) \right| \\ & \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^n - e^{-\frac{1}{2}\tau^2} - \frac{\mu_3}{6\sqrt{n}s^3} (i\tau)^3 e^{-\frac{1}{2}\tau^2} \right| d\tau. \end{aligned} \quad (4.5)$$

**Step 2:** In this step, we show that characteristic function  $\varphi^{a_n}$  of  $\bar{\pi}^{a_n}(x)$  satisfies

$$\sup_{a_n \in \mathbb{R}^+} \int |\varphi^{a_n}(\tau)|^2 d\tau < \infty \quad \text{and} \quad \sup_{a_n \in \mathbb{R}^+, |\tau| \geq \epsilon > 0} |\varphi^{a_n}(\tau)| < 1, \quad (4.6)$$

for any positive  $\epsilon$ .

It is easy to verify that  $r$ -order ( $r \geq 1$ ) moment  $\mu^r$  of  $\pi^{a_n}(x)$  satisfies

$$\mu^r(t) = \frac{d^r \log \Phi(t)}{dt^r} \quad \text{with } t = m^{\leftarrow}(a_n),$$



By Parseval identity

$$\int |\varphi^{a_n}(\tau)|^2 d\tau = 2\pi \int (\bar{\pi}^{a_n}(x))^2 dx \leq 2\pi \sup_{x \in \mathbb{R}} \bar{\pi}^{a_n}(x). \quad (4.7)$$

For the density function  $p(x)$  in (2.1), Theorem 5.4 of Nagaev [9] states that the normalized conjugate density of  $p(x)$ , namely,  $\bar{\pi}^{a_n}(x)$  has the propriety

$$\lim_{a_n \rightarrow \infty} \sup_{x \in \mathbb{R}} |\bar{\pi}^{a_n}(x) - \phi(x)| = 0.$$

Thus for arbitrary positive  $\delta$ , there exists some positive constant  $M$  such that it holds

$$\sup_{a_n \geq M} \sup_{x \in \mathbb{R}} |\bar{\pi}^{a_n}(x) - \phi(x)| \leq \delta,$$

which entails that  $\sup_{a_n \geq M} \sup_{x \in \mathbb{R}} \bar{\pi}^{a_n}(x) < \infty$ . When  $a_n < M$ ,  $\sup_{a_n < M} \sup_{x \in \mathbb{R}} \bar{\pi}^{a_n}(x) < \infty$ ; hence we have

$$\sup_{a_n \in \mathbb{R}^+} \sup_{x \in \mathbb{R}} \bar{\pi}^{a_n}(x) < \infty,$$

which, together with (4.7), gives (4.6). Furthermre,  $\varphi^{a_n}(\tau)$  is not periodic, hence the second inequality of (4.6) holds from Lemma 4 (Chapiter 15, section 1) of [7].

**Step 3:** In this step, we complete the proof by showing that when  $n \rightarrow \infty$

$$\int_{-\infty}^{\infty} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^n - e^{-\frac{1}{2}\tau^2} - \frac{\mu_3}{6\sqrt{n}s^3} (i\tau)^3 e^{-\frac{1}{2}\tau^2} \right| d\tau = o\left(\frac{1}{\sqrt{n}}\right). \quad (4.8)$$

For arbitrarily positive sequence  $a_n$  we have

$$\sup_{a_n \in \mathbb{R}^+} |\varphi^{a_n}(\tau)| = \sup_{a_n \in \mathbb{R}^+} \left| \int_{-\infty}^{\infty} e^{i\tau x} \bar{\pi}^{a_n}(x) dx \right| \leq \sup_{a_n \in \mathbb{R}^+} \int_{-\infty}^{\infty} |e^{i\tau x} \bar{\pi}^{a_n}(x)| dx = 1.$$

In addition,  $\pi^{a_n}(x)$  is integrable, by Riemann-Lebesgue theorem, it holds when  $|\tau| \rightarrow \infty$

$$\sup_{a_n \in \mathbb{R}^+} |\varphi^{a_n}(\tau)| \longrightarrow 0.$$

Thus for any strictly positive  $\omega$ , there exists some corresponding  $N_\omega$  such that if  $|\tau| > \omega$ , it holds

$$\sup_{a_n \in \mathbb{R}^+} |\varphi^{a_n}(\tau)| < N_\omega < 1. \quad (4.9)$$

We now turn to (4.8) which is splitted on  $|\tau| > \omega\sqrt{n}$  and on  $|\tau| \leq \omega\sqrt{n}$ .

It holds

$$\begin{aligned}
& \sqrt{n} \int_{|\tau| > \omega\sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^n - e^{-\frac{1}{2}\tau^2} - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 e^{-\frac{1}{2}\tau^2} \right| d\tau \\
& \leq \sqrt{n} \int_{|\tau| > \omega\sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n})) \right|^n d\tau + \sqrt{n} \int_{|\tau| > \omega\sqrt{n}} \left| e^{-\frac{1}{2}\tau^2} + \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 e^{-\frac{1}{2}\tau^2} \right| d\tau \\
& \leq \sqrt{n} N_\omega^{n-2} \int_{|\tau| > \omega\sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n})) \right|^2 d\tau + \sqrt{n} \int_{|\tau| > \omega\sqrt{n}} e^{-\frac{1}{2}\tau^2} \left( 1 + \left| \frac{\mu_3 \tau^3}{6\sqrt{n}s^3} \right| \right) d\tau. \quad (4.10)
\end{aligned}$$

where the first term of the last line tends to 0 when  $n \rightarrow \infty$ , since

$$\begin{aligned}
& \sqrt{n} N_\omega^{n-2} \int_{|\tau| > \omega\sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n})) \right|^2 d\tau \\
& = \exp \left( \frac{1}{2} \log n + (n-2) \log N_\omega + \log \int_{|\tau| > \omega\sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n})) \right|^2 d\tau \right) \rightarrow 0, \quad (4.11)
\end{aligned}$$

where the last step holds from (4.6) and (4.9). As for the second term of (4.10), by Corollary (3.1), when  $n \rightarrow \infty$ , we have  $|\mu_3/s^3| \rightarrow 0$ . Hence it holds when  $n \rightarrow \infty$

$$\begin{aligned}
& \sqrt{n} \int_{|\tau| > \omega\sqrt{n}} e^{-\frac{1}{2}\tau^2} \left( 1 + \left| \frac{\mu_3 \tau^3}{6\sqrt{n}s^3} \right| \right) d\tau \\
& \leq \sqrt{n} \int_{|\tau| > \omega\sqrt{n}} e^{-\frac{1}{2}\tau^2} |\tau|^3 d\tau = \sqrt{n} \int_{|\tau| > \omega\sqrt{n}} \exp \left\{ -\frac{1}{2}\tau^2 + 3 \log |\tau| \right\} d\tau \\
& = 2\sqrt{n} \exp \left( -\omega^2 n/2 + o(\omega^2 n/2) \right) \rightarrow 0, \quad (4.12)
\end{aligned}$$

where the second equality holds from, for example, Chapter 4 of [1]. (4.10), (4.11) and (4.12) implicate that, when  $n \rightarrow \infty$

$$\int_{|\tau| > \omega\sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^n - e^{-\frac{1}{2}\tau^2} - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 e^{-\frac{1}{2}\tau^2} \right| d\tau = o\left(\frac{1}{\sqrt{n}}\right). \quad (4.13)$$

If  $|\tau| \leq \omega\sqrt{n}$ , it holds

$$\begin{aligned}
& \int_{|\tau| \leq \omega\sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^n - e^{-\frac{1}{2}\tau^2} - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 e^{-\frac{1}{2}\tau^2} \right| d\tau \\
& = \int_{|\tau| \leq \omega\sqrt{n}} e^{-\frac{1}{2}\tau^2} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^n e^{\frac{1}{2}\tau^2} - 1 - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right| d\tau \\
& = \int_{|\tau| \leq \omega\sqrt{n}} e^{-\frac{1}{2}\tau^2} \left| \exp \left\{ n \log \varphi^{a_n}(\tau/\sqrt{n}) + \frac{1}{2}\tau^2 \right\} - 1 - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right| d\tau. \quad (4.14)
\end{aligned}$$

The integrand in the last display is bounded through

$$|e^\alpha - 1 - \beta| = |(e^\alpha - e^\beta) + (e^\beta - 1 - \beta)| \leq (|\alpha - \beta| + \frac{1}{2}\beta^2)e^\gamma, \quad (4.15)$$

where  $\gamma \geq \max(|\alpha|, |\beta|)$ ; this inequality follows replacing  $e^\alpha, e^\beta$  by their power series, for real or complex  $\alpha, \beta$ . Denote by

$$\gamma(\tau) = \log \varphi^{a_n}(\tau) + \frac{1}{2}\tau^2.$$

Since  $\gamma'(0) = \gamma''(0) = 0$ , the third order Taylor expansion of  $\gamma(\tau)$  at  $\tau = 0$  yields

$$\gamma(\tau) = \gamma(0) + \gamma'(0)\tau + \frac{1}{2}\gamma''(0)\tau^2 + \frac{1}{6}\gamma'''(\xi)\tau^3 = \frac{1}{6}\gamma'''(\xi)\tau^3,$$

where  $0 < \xi < \tau$ . Hence it holds

$$\left| \gamma(\tau) - \frac{\mu_3}{6s^3}(i\tau)^3 \right| = \left| \gamma'''(\xi) - \frac{\mu_3}{s_n^3}i^3 \right| \frac{|\tau|^3}{6}.$$

Here  $\gamma'''$  is continuous; thus we can choose  $\omega$  small enough such that  $|\gamma'''(\xi)| < \rho$  for  $|\tau| < \omega$ . Meanwhile, for  $n$  large enough, according to Corollary (3.1), we have  $|\mu_3/s^3| \rightarrow 0$ . Hence it holds for  $n$  large enough

$$\left| \gamma(\tau) - \frac{\mu_3}{6s^3}(i\tau)^3 \right| \leq \left( |\gamma'''(\xi)| + \rho \right) \frac{|\tau|^3}{6} < \rho|\tau|^3. \quad (4.16)$$

Choose  $\omega$  small enough, such that for  $n$  large enough it holds for  $|\tau| < \omega$

$$\left| \frac{\mu_3}{6s^3}(i\tau)^3 \right| \leq \frac{1}{4}\tau^2, \quad |\gamma(\tau)| \leq \frac{1}{4}\tau^2.$$

For this choice of  $\omega$ , when  $|\tau| < \omega$  we have

$$\max \left( \left| \frac{\mu_3}{6s^3}(i\tau)^3 \right|, |\gamma(\tau)| \right) \leq \frac{1}{4}\tau^2. \quad (4.17)$$

Replacing  $\tau$  by  $\tau/\sqrt{n}$ , it holds for  $|\tau| < \omega\sqrt{n}$

$$\begin{aligned} & \left| n \log \varphi^{a_n}(\tau/\sqrt{n}) + \frac{1}{2}\tau^2 - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right| \\ &= n \left| \log \varphi^{a_n}(\tau/\sqrt{n}) + \frac{1}{2}\left(\frac{\tau}{\sqrt{n}}\right)^2 - \frac{\mu_3}{6s^3}\left(\frac{i\tau}{\sqrt{n}}\right)^3 \right| \\ &= n \left| \gamma\left(\frac{\tau}{\sqrt{n}}\right) - \frac{\mu_3}{6s^3}\left(\frac{i\tau}{\sqrt{n}}\right)^3 \right| < \frac{\rho|\tau|^3}{\sqrt{n}}, \end{aligned} \quad (4.18)$$

where the last inequality holds from (4.16). In a similar way, with (4.17), it also holds for  $|\tau| < \omega\sqrt{n}$

$$\begin{aligned} & \max \left( \left| n \log \varphi^{a_n}(\tau/\sqrt{n}) + \frac{1}{2}\tau^2 \right|, \left| \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right| \right) \\ &= n \max \left( \left| \gamma\left(\frac{\tau}{\sqrt{n}}\right) \right|, \left| \frac{\mu_3}{6s^3}\left(\frac{i\tau}{\sqrt{n}}\right)^3 \right| \right) \leq \frac{1}{4}\tau^2. \end{aligned} \quad (4.19)$$

Apply (4.15) to estimate the integrand of last line of (4.14), with the choice of  $\omega$  in (4.16) and (4.17), using (4.18) and (4.19) we have for  $|\tau| < \omega\sqrt{n}$

$$\begin{aligned}
& \left| \exp \left\{ n \log \varphi^{a_n}(\tau/\sqrt{n}) + \frac{1}{2}\tau^2 \right\} - 1 - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right| \\
& \leq \left( \left| n \log \varphi^{a_n}(\tau/\sqrt{n}) + \frac{1}{2}\tau^2 - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right| + \frac{1}{2} \left| \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right|^2 \right) \\
& \quad \times \exp \left[ \max \left( \left| n \log \varphi^{a_n}(\tau/\sqrt{n}) + \frac{1}{2}\tau^2 \right|, \left| \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right| \right) \right] \\
& \leq \left( \frac{\rho|\tau|^3}{\sqrt{n}} + \frac{1}{2} \left| \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right|^2 \right) \exp \left( \frac{\tau^2}{4} \right) \\
& = \left( \frac{\rho|\tau|^3}{\sqrt{n}} + \frac{\mu_3^2\tau^6}{72ns^6} \right) \exp \left( \frac{\tau^2}{4} \right).
\end{aligned}$$

Use this upper bound to (4.14), we obtain

$$\begin{aligned}
& \int_{|\tau| \leq \omega\sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^n - e^{-\frac{1}{2}\tau^2} - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 e^{-\frac{1}{2}\tau^2} \right| d\tau \\
& \leq \int_{|\tau| \leq \omega\sqrt{n}} \exp \left( -\frac{\tau^2}{4} \right) \left( \frac{\rho|\tau|^3}{\sqrt{n}} + \frac{\mu_3^2\tau^6}{72ns^6} \right) d\tau \\
& = \frac{\rho}{\sqrt{n}} \int_{|\tau| \leq \omega\sqrt{n}} \exp \left( -\frac{\tau^2}{4} \right) |\tau|^3 d\tau + \frac{\mu_3^2}{72ns^6} \int_{|\tau| \leq \omega\sqrt{n}} \exp \left( -\frac{\tau^2}{4} \right) \tau^6 d\tau,
\end{aligned}$$

where both the first integral and the second integral are finite, and  $\rho$  is arbitrarily small; additionally, by Corollary (3.1),  $\mu_3^2/s^6 \rightarrow 0$  when  $n$  large enough, hence it holds when  $n \rightarrow \infty$

$$\int_{|\tau| \leq \omega\sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^n - e^{-\frac{1}{2}\tau^2} - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 e^{-\frac{1}{2}\tau^2} \right| d\tau = o\left(\frac{1}{\sqrt{n}}\right). \quad (4.20)$$

Now (4.13) and (4.20) give (4.8). Further, coming back to (4.5), using (4.8), we obtain

$$\left| \bar{\pi}_n^{a_n}(x) - \phi(x) - \frac{\mu_3}{6\sqrt{n}s^3}(x^3 - 3x)\phi(x) \right| = o\left(\frac{1}{\sqrt{n}}\right),$$

which concludes the proof.

## 5 Gibbs' conditional principles under extreme events

We now explore Gibbs conditional principles under extreme events. The first result is a pointwise approximation of the conditional density  $p_{a_n}(y_1^k)$  on  $\mathbb{R}^k$  for fixed  $k$ . As a by-product we also address the local approximation of  $p_{A_n}$  where

$$p_{A_n}(y_1^k) := p(X_1^k = y_1^k | S_1^n > na_n).$$

However these local approximations are of poor interest when comparing  $p_{a_n}$  to its approximation.

We consider the case  $k = 1$ . For  $Y_1^n$  a random vector with density  $p_{a_n}$  we first provide a density  $g_{a_n}$  on  $\mathbb{R}$  such that

$$p_{a_n}(Y_1) = g_{a_n}(Y_1)(1 + R_n)$$

where  $R_n$  is a function of the vector  $Y_1^n$  which goes to 0 as  $n$  tends to infinity. The above statement may also be written as

$$p_{a_n}(y_1) = g_{a_n}(y_1)(1 + o_{P_{a_n}}(1)) \quad (5.1)$$

where  $P_{a_n}$  is the joint probability measure of the vector  $Y_1^n$  under the condition  $(S_1^n = na_n)$ . This statement is of a different nature with respect to the above one, since it amounts to prove the approximation on typical realisations under the conditional sampling scheme. We will deduce from (5.1) that the  $L^1$  distance between  $p_{a_n}$  and  $g_{a_n}$  goes to 0 as  $n$  tends to infinity. It would be interesting to extend these results to the case when  $k = k_n$  is close to  $n$ , as done in [4] in all cases from the CLT to the LDP ranges. The extreme deviation case is more involved, which led us to restrict this study to the case when  $k = 1$  (or  $k$  fixed, similarly).

## 5.1 A local result in $\mathbb{R}^k$

Fix  $y_1^k := (y_1, \dots, y_k)$  in  $\mathbb{R}^k$  and define  $s_i^j := y_i + \dots + y_j$  for  $1 \leq i < j \leq k$ .

Define  $t_i$  through

$$m(t_i) := \frac{na_n - s_1^i}{n - i}. \quad (5.2)$$

For the sake of brevity, we write  $m_i$  instead of  $m(t_i)$ , and define  $s_i^2 := s^2(t_i)$ . We have the following conditional density.

Consider the following condition

$$\lim_{n \rightarrow \infty} \frac{\psi(t_k)^2}{\sqrt{n\psi'(t_k)}} = 0, \quad (5.3)$$

which can be seen as a growth condition on  $a_n$ , avoiding too large increases of this sequence.

For  $0 \leq i \leq k - 1 < n$ , define  $z_i$  through

$$z_i = \frac{m_i - y_{i+1}}{s_i \sqrt{n - i - 1}}.$$

**Lemma 5.1.** Assume that  $p(x)$  satisfies (2.1) and  $h(x) \in \mathfrak{R}$ . Let  $t_i$  is defined in (5.2). Assume that  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$  and that (5.3) holds. then it holds as  $a_n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \sup_{0 \leq i \leq k-1} z_i = 0.$$

Proof: When  $n \rightarrow \infty$ , it holds

$$z_i \sim m_i / s_i \sqrt{n-i-1} \sim m_i / (s_i \sqrt{n}).$$

From Theorem 3.1, it holds  $m(t) \sim \psi(t)$  and  $s(t) \sim \sqrt{\psi'(t)}$ . Hence we have

$$z_i \sim \frac{\psi(t_i)}{\sqrt{n\psi'(t_i)}}. \quad (5.4)$$

By (5.2),  $m_i \sim m_k$  as  $n \rightarrow \infty$ . Consider  $m_k \sim \psi(t_k)$ . Then it holds

$$m_i \sim \psi(t_k).$$

In addition,  $m_i \sim \psi(t_i)$  by Theorem 3.1, this implies it holds

$$\psi(t_i) \sim \psi(t_k). \quad (5.5)$$

**Case 1:** if  $h(x) \in R_\beta$ . We have  $h(x) = x^\beta l_0(x)$ ,  $l_0(x) \in R_0$ ,  $\beta > 0$ . Hence

$$h'(x) = x^{\beta-1} l_0(x) (\beta + \epsilon(x)),$$

set  $x = \psi(t)$ , we get

$$h'(\psi(t)) = (\psi(t))^{\beta-1} l_0(\psi(t)) (\beta + \epsilon(\psi(t))). \quad (5.6)$$

Notice that it holds  $\psi'(t) = 1/h'(\psi(t))$ , combine (5.5) with (5.6), we obtain

$$\frac{\psi'(t_i)}{\psi'(t_k)} = \frac{h'(\psi(t_k))}{h'(\psi(t_i))} = \frac{(\psi(t_k))^{\beta-1} l_0(\psi(t_k)) (\beta + \epsilon(\psi(t_k)))}{(\psi(t_i))^{\beta-1} l_0(\psi(t_i)) (\beta + \epsilon(\psi(t_i)))} \rightarrow 1,$$

where we use the slowly varying propriety of  $l_0$ . Thus it holds

$$\psi'(t_i) \sim \psi'(t_k),$$

which, together with (5.5), is put into (5.4) to yield

$$z_i \sim \frac{\psi(t_k)}{\sqrt{n\psi'(t_k)}}.$$

Hence we have under condition (5.3)

$$z_i^2 \sim \frac{\psi(t_k)^2}{n\psi'(t_k)} = \frac{\psi(t_k)^2}{\sqrt{n}\psi'(t_k)} \frac{1}{\sqrt{n}} = o\left(\frac{1}{\sqrt{n}}\right),$$

which implies further  $z_i \rightarrow 0$ . Note that the final step is used in order to relax the strength of the growth condition on  $a_n$ .

**Case 2:** if  $h(x) \in R_\infty$ . By (5.2), it holds  $m(t_k) \geq m(t_i)$  as  $n \rightarrow \infty$ . Since the function  $t \rightarrow m(t)$  is increasing, we have

$$t_i \leq t_k.$$

The function  $t \rightarrow \psi'(t)$  is decreasing, since

$$\psi''(t) = -\frac{\psi(t)}{t^2}\epsilon(t)(1 + o(1)) < 0 \quad \text{as } t \rightarrow \infty.$$

Therefore it holds as  $n \rightarrow \infty$

$$\psi'(t_i) \geq \psi'(t_k),$$

which, combined with (5.4) and (5.5), yields

$$z_i \sim \frac{\psi(t_k)}{\sqrt{n\psi'(t_i)}} \leq \frac{2\psi(t_k)}{\sqrt{n\psi'(t_k)}},$$

hence we have

$$z_i^2 \leq \frac{4\psi(t_k)^2}{n\psi'(t_k)} = \frac{4\psi(t_k)^2}{\sqrt{n}\psi'(t_k)} \frac{1}{\sqrt{n}} = o\left(\frac{1}{\sqrt{n}}\right),$$

where the last step holds from condition (5.3). Further it holds  $z_i \rightarrow 0$ .

**Theorem 5.1.** *With the above notation and hypotheses, assuming (5.3), it holds*

$$p_{a_n}(y_1^k) = p(X_1^k = y_1^k | S_1^n = na_n) = g_m(y_1^k) \left(1 + o(1)\right).$$

with

$$g_m(y_1^k) = \prod_{i=0}^{k-1} \left( \pi^{m_i}(X_{i+1} = y_{i+1}) \right).$$

Proof:

Using Bayes formula,

$$\begin{aligned} p_{a_n}(y_1^k) &:= p(X_1^k = y_1^k | S_1^n = na_n) = p(X_1 = y_1 | S_1^n = na_n) \prod_{i=1}^{k-1} p(X_{i+1} = y_{i+1} | X_1^i = y_1^i, S_1^n = na_n) \\ &= \prod_{i=0}^{k-1} p(X_{i+1} = y_{i+1} | S_{i+1}^n = na_n - s_1^i). \end{aligned} \quad (5.7)$$

We make use of the following invariance property: For all  $y_1^k$  and all  $\alpha > 0$

$$p(X_{i+1} = y_{i+1} | X_1^i = y_1^i, S_1^n = na_n) = \pi^\alpha(X_{i+1} = y_{i+1} | X_1^i = y_1^i, S_1^n = na_n)$$

where on the LHS, the r.v.'s  $X_1^i$  are sampled i.i.d. under  $p$  and on the RHS, sampled i.i.d. under  $\pi^\alpha$ . *It thus holds*

$$\begin{aligned} p(X_{i+1} = y_{i+1} | S_{i+1}^n = na_n - S_1^i) &= \pi^{m_i}(X_{i+1} = y_{i+1} | S_{i+1}^n = na_n - s_1^i) \\ &= \pi^{m_i}(X_{i+1} = y_{i+1}) \frac{\pi^{m_i}(S_{i+2}^n = na_n - s_1^{i+1})}{\pi^{m_i}(S_{i+1}^n = na_n - s_1^i)} \\ &= \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi^{m_i}(X_{i+1} = y_{i+1}) \frac{\widetilde{\pi_{n-i-1}}(\frac{m_i - y_{i+1}}{s_i \sqrt{n-i-1}})}{\widetilde{\pi_{n-i}}(0)}, \end{aligned} \quad (5.8)$$

where  $\widetilde{\pi_{n-i-1}}$  is the normalized density of  $S_{i+2}^n$  under i.i.d. sampling under  $\pi^{m_i}$ ; correspondingly,  $\widetilde{\pi_{n-i}}$  is the normalized density of  $S_{i+1}^n$  under the same sampling. Note that a r.v. with density  $\pi^{m_i}$  has expectation  $m_i$  and variance  $s_i^2$ .

Write  $z_i = \frac{m_i - y_{i+1}}{s_i \sqrt{n-i-1}}$ , and perform a third-order Edgeworth expansion of  $\widetilde{\pi_{n-i-1}}(z_i)$ , using Theorem 4.1. It follows

$$\widetilde{\pi_{n-i-1}}(z_i) = \phi(z_i) \left( 1 + \frac{\mu_3^i}{6s_i^3 \sqrt{n-1}} (z_i^3 - 3z_i) \right) + o\left(\frac{1}{\sqrt{n}}\right), \quad (5.9)$$

The approximation of  $\widetilde{\pi_{n-i}}(0)$  is obtained from (5.9)

$$\widetilde{\pi_{n-i}}(0) = \phi(0) \left( 1 + o\left(\frac{1}{\sqrt{n}}\right) \right). \quad (5.10)$$

Put (5.9) and (5.10) into (5.8) to obtain

$$\begin{aligned} p(X_{i+1} = y_{i+1} | S_{i+1}^n = na_n - S_1^i) &= \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi^{m_i}(X_{i+1} = y_{i+1}) \frac{\phi(z_i)}{\phi(0)} \left[ 1 + \frac{\mu_3^i}{6s_i^3 \sqrt{n-1}} (z_i^3 - 3z_i) + o\left(\frac{1}{\sqrt{n}}\right) \right] \\ &= \frac{\sqrt{2\pi(n-i)}}{\sqrt{n-i-1}} \pi^{m_i}(X_{i+1} = y_{i+1}) \phi(z_i) (1 + R_n + o(1/\sqrt{n})), \end{aligned} \quad (5.11)$$

where

$$R_n = \frac{\mu_3^i}{6s_i^3 \sqrt{n-1}} (z_i^3 - 3z_i).$$

Under condition (5.3), using Lemma 5.1, it holds  $z_i \rightarrow 0$  as  $a_n \rightarrow \infty$ , and under Corollary (3.1),  $\mu_3^i/s_i^3 \rightarrow 0$ . This yields

$$R_n = o(1/\sqrt{n}),$$



which, combined with (5.11), gives

$$\begin{aligned} p(X_{i+1} = y_{i+1} | s_{i+1}^n = na_n - S_1^i) &= \frac{\sqrt{2\pi(n-i)}}{\sqrt{n-i-1}} \pi^{m_i}(X_{i+1} = y_{i+1}) \phi(z_i) (1 + o(1/\sqrt{n})) \\ &= \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi^{m_i}(X_{i+1} = y_{i+1}) (1 - z_i^2/2 + o(z_i^2)) (1 + o(1/\sqrt{n})), \end{aligned}$$

where we use one Taylor expansion in second equality. Using once more Lemma 5.1, under conditions (5.3), we have as  $a_n \rightarrow \infty$

$$z_i^2 = o(1/\sqrt{n}),$$

hence we get

$$p(X_{i+1} = y_{i+1} | S_{i+1}^n = na_n - s_1^i) = \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi^{m_i}(X_{i+1} = y_{i+1}) (1 + o(1/\sqrt{n})),$$

which together with (5.7) yields

$$\begin{aligned} p(X_1^k = y_1^k | S_1^n = na_n) &= \prod_{i=0}^{k-1} \left( \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi^{m_i}(X_{i+1} = y_{i+1}) (1 + o(1/\sqrt{n})) \right) \\ &= \prod_{i=0}^{k-1} \left( \pi^{m_i}(X_{i+1} = y_{i+1}) \right) \prod_{i=0}^{k-1} \left( \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \right) \prod_{i=0}^{k-1} \left( 1 + o\left(\frac{1}{\sqrt{n}}\right) \right) \\ &= \left( 1 + o\left(\frac{1}{\sqrt{n}}\right) \right) \prod_{i=0}^{k-1} \left( \pi^{m_i}(X_{i+1} = y_{i+1}) \right), \end{aligned}$$

The proof is completed.

Define  $t$  through  $m(t) = a_n$ , replace condition (5.3) by

$$\lim_{n \rightarrow \infty} \frac{\psi(t)^2}{\sqrt{n\psi'(t)}} = 0, \tag{5.12}$$

then for fixed  $k$ , an equivalent statement is

**Theorem 5.2.** *Under the same hypotheses as in the previous Theorem*

$$p_{a_n}(y_1^k) = p(X_1^k = y_1^k | S_1^n = na_n) = g_{a_n}(y_1^k) \left( 1 + o\left(\frac{1}{\sqrt{n}}\right) \right).$$

with

$$g_{a_n}(y_1^k) = \prod_{i=1}^k \left( \pi^{a_n}(X_i = y_i) \right).$$

Proof:

Using the notations of Theorem 5.1, by (5.7), we obtain

$$p(X_1^k = y_1^k | S_1^n = na_n) = \prod_{i=0}^{k-1} p(X_{i+1} = y_{i+1} | S_{i+1}^n = na_n - S_1^i). \quad (5.13)$$

(5.8) is replaced by

$$p(X_{i+1} = y_{i+1} | S_{i+1}^n = na_n - S_1^i) = \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi^{a_n}(X_{i+1} = y_{i+1}) \frac{\widetilde{\pi_{n-i-1}^{a_n}}\left(\frac{(i+1)a_n - S_1^{i+1}}{s\sqrt{n-i-1}}\right)}{\widetilde{\pi_{n-i}^{a_n}}\left(\frac{ia_n - S_1^i}{s\sqrt{n-i}}\right)}, \quad (5.14)$$

where  $\widetilde{\pi_{n-i-1}^{a_n}}((i+1)a_n - y_{i+1}/s_i\sqrt{n-i-1})$  is the normalized density of  $\pi^{a_n}(S_{i+2}^n = na_n - S_1^{i+1})$ , and  $\pi^{a_n}$  has the expectation  $a_n$  and variance  $s$ . Correspondingly,  $\widetilde{\pi_{n-i}^{a_n}}((ia_n - S_1^i)/s\sqrt{n-i})$  is the normalized density of  $\pi^{a_n}(S_{i+1}^n = na_n - S_1^i)$ .

Write  $z_i = \frac{(i+1)a_n - S_1^{i+1}}{s\sqrt{n-i-1}}$ , by Theorem 4.1 one three-order Edgeworth expansion yields

$$\widetilde{\pi_{n-i-1}^{a_n}}(z_i) = \phi(z_i) \left(1 + R_n^i\right) + o\left(\frac{1}{\sqrt{n}}\right), \quad (5.15)$$

where

$$R_n^i = \frac{\mu_3}{6s^3\sqrt{n-1}}(z_i^3 - 3z_i).$$

Set  $i = i - 1$ , the approximation of  $\widetilde{\pi_{n-i}^{a_n}}$  is obtained from (5.15)

$$\widetilde{\pi_{n-i}^{a_n}}(z_{i-1}) = \phi(z_{i-1}) \left(1 + R_n^{i+1}\right) + o\left(\frac{1}{\sqrt{n}}\right). \quad (5.16)$$

When  $a_n \rightarrow \infty$ , using Theorem 3.1, it holds

$$\begin{aligned} \sup_{0 \leq i \leq k-1} z_i^2 &\sim \frac{(i+1)^2 a_n^2}{s^2 n} \leq \frac{2k^2 a_n^2}{s^2 n} = \frac{2k^2 (m(t))^2}{s^2 n} \\ &\sim \frac{2k^2 (\psi(t))^2}{\psi'(t)n} = \frac{2k^2 (\psi(t))^2}{\sqrt{n}\psi'(t)} \frac{1}{\sqrt{n}} = o\left(\frac{1}{\sqrt{n}}\right), \end{aligned} \quad (5.17)$$

where last step holds under condition (5.12). Hence it holds  $z_i \rightarrow 0$  uniformly in  $i$  as  $a_n \rightarrow \infty$ , and by Corollary (3.1),  $\mu_3/s^3 \rightarrow 0$ , then it follows

$$R_n^i = o(1/\sqrt{n}) \quad R_n^{i+1} = o(1/\sqrt{n}),$$

then put (5.15) and (5.16) into (5.14), we obtain

$$\begin{aligned} p(X_{i+1} = y_{i+1} | S_{i+1}^n = na_n - S_1^i) &= \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi^{a_n}(X_{i+1} = y_{i+1}) \frac{\phi(z_i)}{\phi(z_{i-1})} (1 + o(1/\sqrt{n})) \\ &= \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi^{m_i}(X_{i+1} = y_{i+1}) (1 - (z_i^2 - z_{i-1}^2)/2 + o(z_i^2 - z_{i-1}^2)) (1 + o(1/\sqrt{n})), \end{aligned}$$

where we use one Taylor expansion in second equality. Using (5.17), we have as  $a_n \rightarrow \infty$

$$|z_i^2 - z_{i-1}^2| = o(1/\sqrt{n}),$$

hence we get

$$p(X_{i+1} = y_{i+1} | S_{i+1}^n = na_n - S_1^i) = \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi^{a_n}(X_{i+1} = y_{i+1}) (1 + o(1/\sqrt{n})),$$

which together with (5.13) yields

$$\begin{aligned} p(X_1^k = y_1^k | S_1^n = na_n) &= \prod_{i=0}^{k-1} \left( \pi^{a_n}(X_{i+1} = y_{i+1}) \sqrt{\frac{n}{n-k}} \right) \prod_{i=0}^{k-1} \left( 1 + o\left(\frac{1}{\sqrt{n}}\right) \right) \\ &= \left( 1 + o\left(\frac{1}{\sqrt{n}}\right) \right) \prod_{i=0}^{k-1} \left( \pi^{a_n}(X_{i+1} = y_{i+1}) \right). \end{aligned} \quad (5.18)$$

This completes the proof.

**Remark 5.1.** *The above result shows that asymptotically the point condition ( $S_1^n = na_n$ ) leaves blocks of  $k$  of the  $X_i$ 's independent. Obviously this property does not hold for large values of  $k$ , close to  $n$ . A similar statement holds in the LDP range, conditioning either on ( $S_1^n = na$ ) (see Diaconis and Friedman 1988), or on ( $S_1^n \geq na$ ); see Csiszar 1984 for a general statement on asymptotic conditional independence.*

Using the same proof as in Theorem (5.2), we obtain the following corollary.

**Corollary 5.1.** *It holds*

$$p_a(y_1^k) = p(X_1^k = y_1^k | S_1^n = na_n) = g_a(y_1^k) \left( 1 + o\left(\frac{k}{\sqrt{n}}\right) \right).$$

with

$$g_a(y_1^k) = \prod_{i=1}^k \left( \pi^a(X_i = y_i) \right).$$

## 5.2 Strengthening of the local Gibbs conditional principle

We now turn to a stronger approximation of  $p_{a_n}$ . Consider  $Y_1^n$  with density  $p_{a_n}$  and the resulting random variable  $p_{a_n}(Y_1)$ . We prove the following result

**Theorem 5.3.** *With all the above notation and hypotheses it holds*

$$p_{a_n}(Y_1) = g_{a_n}(Y_1)(1 + R_n)$$

where

$$g_{a_n} = \pi^{a_n}$$

the tilted density at point  $a_n$ , and where  $R_n$  is a function of  $Y_1^n$  such that  $P_{a_n}(|R_n| > \delta\sqrt{n}) \rightarrow 0$  as  $n \rightarrow \infty$  for any positive  $\delta$ .

This result is of much greater relevance than the previous ones. Indeed under  $P_{a_n}$  the r.v.  $Y_1$  may take large values. At the contrary simple approximation of  $p_{a_n}$  by  $g_{a_n}$  on  $\mathbb{R}_+$  only provides some knowledge on  $p_{a_n}$  on sets with smaller and smaller probability under  $p_{a_n}$ . Also it will be proved that as a consequence of the above result, the  $L^1$  norm between  $p_{a_n}$  and  $g_{a_n}$  goes to 0 as  $n \rightarrow \infty$ , a result out of reach through the aforementioned results.

In order to adapt the proof of Theorem \*\*\* to the present setting it is necessary to get some insight on the plausible values of  $Y_1$  under  $P_{a_n}$ . It holds

**Lemma 5.2.** *Under  $P_{a_n}$  it holds*

$$Y_1 = O_{P_{a_n}}(a_n)$$

Proof: This is a consequence of Markov Inequality:

$$P(Y_1 > u | S_1^n = na_n) \leq \frac{E(Y_1 | S_1^n = na_n)}{u} = \frac{a_n}{u}$$

which goes to 0 for all  $u = u_n$  such that  $\lim_{n \rightarrow \infty} u_n/a_n = \infty$ .

We now turn back to the proof of our result, replacing  $y_1^k$  by  $Y_1$  in (5.14).

It holds

$$P(X_1 = Y_1 | S_1^n = na_n) = P(X_1 = Y_1) \frac{P(S_2^n = na_n - Y_1)}{P(S_1^n = na_n)}$$

in which the tilting substitution of measures is performed, with tilting density  $\pi^{a_n}$ , followed by normalization. Now if the growth condition (5.3) holds, namely

$$\lim_{n \rightarrow \infty} \frac{\psi(t)}{\sqrt{n\psi'(t)}} = 0$$

with  $m(t) = a_n$  it follows that

$$P(X_1 = Y_1 | S_1^n = na_n) = \pi^{a_n}(Y_1)(1 + R_n)$$

as claimed where the order of magnitude of  $R_n$  is  $o_{P_{a_n}}(1/\sqrt{n})$ . We have proved Theorem 5.3.

Denote the conditional probabilities by  $P_{a_n}$  and  $G_{a_n}$  which correspond to the density functions  $p_{a_n}$  and  $g_{a_n}$ , respectively.

### 5.3 Gibbs principle in variation norm

We now consider the approximation of  $P_{a_n}$  by  $G_{a_n}$  in variation norm.

The main ingredient is the fact that in the present setting approximation of  $p_{a_n}$  by  $g_{a_n}$  in probability plus some rate implies approximation of the corresponding measures in variation norm. This approach has been developped in Broniatowski and Caron (2012); we state a first lemma which states that wether two densities are equivalent in probability with small relative error when measured according to the first one, then the same holds under the sampling of the second.

Let  $\mathfrak{R}_n$  and  $\mathfrak{S}_n$  denote two p.m's on  $\mathbb{R}^n$  with respective densities  $\mathfrak{r}_n$  and  $\mathfrak{s}_n$ .

**Lemma 5.3.** *Suppose that for some sequence  $\varepsilon_n$  which tends to 0 as  $n$  tends to infinity*

$$\mathfrak{r}_n(Y_1^n) = \mathfrak{s}_n(Y_1^n)(1 + o_{\mathfrak{R}_n}(\varepsilon_n)) \quad (5.19)$$

*as  $n$  tends to  $\infty$ . Then*

$$\mathfrak{s}_n(Y_1^n) = \mathfrak{r}_n(Y_1^n)(1 + o_{\mathfrak{S}_n}(\varepsilon_n)). \quad (5.20)$$

*Proof.* Denote

$$A_{n,\varepsilon_n} := \{y_1^n : (1 - \varepsilon_n)\mathfrak{s}_n(y_1^n) \leq \mathfrak{r}_n(y_1^n) \leq \mathfrak{s}_n(y_1^n)(1 + \varepsilon_n)\}.$$

It holds for all positive  $\delta$

$$\lim_{n \rightarrow \infty} \mathfrak{R}_n(A_{n,\delta\varepsilon_n}) = 1.$$

Write

$$\mathfrak{R}_n(A_{n,\delta\varepsilon_n}) = \int \mathbf{1}_{A_{n,\delta\varepsilon_n}}(y_1^n) \frac{\mathfrak{r}_n(y_1^n)}{\mathfrak{s}_n(y_1^n)} \mathfrak{s}_n(y_1^n) dy_1^n.$$

Since

$$\mathfrak{R}_n(A_{n,\delta\varepsilon_n}) \leq (1 + \delta\varepsilon_n)\mathfrak{S}_n(A_{n,\delta\varepsilon_n})$$

it follows that

$$\lim_{n \rightarrow \infty} \mathfrak{S}_n(A_{n,\delta\varepsilon_n}) = 1,$$

which proves the claim. □

Applying this Lemma to the present setting yields

$$g_{a_n}(Y_1) = p_{a_n}(Y_1) (1 + o_{G_{a_n}}(1/\sqrt{n}))$$

as  $n \rightarrow \infty$ .

This fact entails, as in [4]

**Theorem 5.4.** *Under all the notation and hypotheses above the total variation norm between  $P_{a_n}$  and  $G_{a_n}$  goes to 0 as  $n \rightarrow \infty$ .*

The proof goes as follows

For all  $\delta > 0$ , let

$$E_\delta := \left\{ y \in \mathbb{R} : \left| \frac{p_{a_n}(y) - g_{a_n}(y)}{g_{a_n}(y)} \right| < \delta \right\}$$

which

$$\lim_{n \rightarrow \infty} P_{a_n}(E_\delta) = \lim_{n \rightarrow \infty} G_{a_n}(E_\delta) = 1. \quad (5.21)$$

It holds

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P_{a_n}(C \cap E_\delta) - G_{a_n}(C \cap E_\delta)| \leq \delta \sup_{C \in \mathcal{B}(\mathbb{R})} \int_{C \cap E_\delta} g_{a_n}(y) dy \leq \delta.$$

By the above result (5.21)

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P_{a_n}(C \cap E_\delta) - P_{a_n}(C)| < \eta_n$$

and

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |G_{a_n}(C \cap E_\delta) - G_{a_n}(C)| < \eta_n$$

for some sequence  $\eta_n \rightarrow 0$  ; hence

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P_{a_n}(C) - G_{a_n}(C)| < \delta + 2\eta_n$$

for all positive  $\delta$ , which proves the claim.

As a consequence, applying Scheffé's Lemma

$$\int |p_{a_n} - g_{a_n}| dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

**Remark 5.2.** *This result is to be paralleled with Theorem 1.6 in Diaconis and Freedman [6] and Theorem 2.15 in Dembo and Zeitouni [5] which provide a rate for this convergence in the LDP range.*

## 5.4 The asymptotic location of $X$ under the conditioned distribution

This section intends to provide some insight on the behaviour of  $X_1$  under the condition  $(S_1^n = na_n)$ ; this will be extended further on to the case when  $(S_1^n \geq na_n)$  and to be considered in parallel with similar facts developped in [4] for larger values of  $a_n$ .

It will be seen that conditionally on  $(S_1^n = na_n)$  the marginal distribution of the sample concentrates around  $a_n$ . Let  $\mathcal{X}_t$  be a r.v. with density  $\pi^{a_n}$  where  $m(t) = a_n$  and  $a_n$  satisfies (5.3). Recall that  $E\mathcal{X}_t = a_n$  and  $\text{Var}\mathcal{X}_t = s^2$ . We evaluate the moment generating function of the normalized variable  $(\mathcal{X}_t - a_n)/s$ . It holds

$$\log E \exp \lambda (\mathcal{X}_t - a_n) / s = -\lambda a_n / s + \log \Phi \left( t + \frac{\lambda}{s} \right) - \log \Phi(t).$$

A second order Taylor expansion in the above display yields

$$\log E \exp \lambda (\mathcal{X}_t - a_n) / s = \frac{\lambda^2 s^2 \left( t + \frac{\theta \lambda}{s} \right)}{2 s^2}$$

where  $\theta = \theta(t, \lambda) \in (0, 1)$ . It holds

**Lemma 5.4.** *Under the above hypotheses and notation, for any compact set  $K$ ,*

$$\lim_{n \rightarrow \infty} \sup_{u \in K} \frac{s^2 \left( t + \frac{u}{s} \right)}{s^2} = 1.$$

Proof: **Case 1:** if  $h(t) \in R_\beta$ . By Theorem 3.1, it holds  $s^2 \sim \psi'(t)$  with  $\psi(t) \sim t^{1/\beta} l_1(t)$ , where  $l(t)$  is some slowly varying function. And we have also  $\psi'(t) = 1/h'(\psi(t))$ , hence by (??) it follows

$$\begin{aligned} \frac{1}{s^2} &\sim h'(\psi(t)) = \psi(t)^{\beta-1} l_0(\psi(t)) (\beta + \epsilon(\psi(t))) \\ &\sim \beta t^{1-1/\beta} l_1(t)^{\beta-1} l_0(\psi(t)) = o(t), \end{aligned}$$

which implies that for any  $u \in K$  it holds

$$\frac{u}{s} = o(t),$$

$$\begin{aligned} \frac{s^2 (t + u/s)}{s^2} &\sim \frac{\psi'(t + u/s)}{\psi'(t)} = \frac{\psi(t)^{\beta-1} l_0(\psi(t)) (\beta + \epsilon(\psi(t)))}{(\psi(t + u/s))^{\beta-1} l_0(\psi(t + u/s)) (\beta + \epsilon(\psi(t + u/s)))} \\ &\sim \frac{\psi(t)^{\beta-1}}{\psi(t + u/s)^{\beta-1}} \sim \frac{t^{1-1/\beta} l_1(t)^{\beta-1}}{(t + u/s)^{1-1/\beta} l_1(t + u/s)^{\beta-1}} \rightarrow 1. \end{aligned}$$

**Case 2:** if  $h(t) \in R_\infty$ . Then we have in this case  $\psi(t) \in \widetilde{R}_0$ , hence it holds

$$\frac{1}{st} \sim \frac{1}{t\sqrt{\psi'(t)}} = \sqrt{\frac{1}{t\psi(t)\epsilon(t)}} \longrightarrow 0,$$

which last step holds from condition (2.7). Hence for any  $u \in K$ , we get as  $n \rightarrow \infty$

$$\frac{u}{s} = o(t),$$

thus using the slowly varying propriety of  $\psi(t)$  we have

$$\begin{aligned} \frac{s^2(t+u/s)}{s^2} &\sim \frac{\psi'(t+u/s)}{\psi'(t)} = \frac{\psi(t+u/s)\epsilon(t+u/s)}{t+u/s} \frac{t}{\psi(t)\epsilon(t)} \\ &\sim \frac{\epsilon(t+u/s)}{\epsilon(t)} = \frac{\epsilon(t) + O(\epsilon'(t)u/s)}{\epsilon(t)} \longrightarrow 1, \end{aligned}$$

where we use one Taylor expansion in the second line, and last step holds from condition (2.6). This completes the proof.

Applying the above Lemma it follows that the normalized r.v's  $(\mathcal{X}_t - a_n)/s$  converge to a standard normal variable  $N(0, 1)$  in distribution, as  $n \rightarrow \infty$ . This amount to say that

$$\mathcal{X}_t = a_n + sN(0, 1) + o_{\Pi^{a_n}}(1).$$

Recall that  $\lim_{n \rightarrow \infty} s = 0$ , which implies that  $\mathcal{X}_t$  concentrates around  $a_n$  with rate  $s$ . Due to Theorem 5.4 the same holds for  $X_1$  under  $(S_1^n = na_n)$ .

## 5.5 Differences between Gibbs principle under LDP and under extreme deviation

It is of interest to confront the present results with the general form of the Gibbs principle under linear constraints in the LDP range. We recall briefly and somehow informally the main classical facts in a simple setting similar as the one used in this paper.

Let  $X_1, \dots, X_n$  denote  $n$  i.i.d. real valued r.v's with distribution  $P$  and density  $p$  and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a measurable function such that  $\Phi_f(\lambda) := E \exp \lambda f(X_1)$  is finite for  $\lambda$  in a non void neighborhood of 0 (the so-called Cramer condition). Denote  $m_f(\lambda)$  and  $s_f^2(\lambda)$  the first and second derivatives of  $\log \Phi_f(\lambda)$ . Consider the point set condition  $E_n := (\frac{1}{n} \sum_{i=1}^n f(X_i) = 0)$  and let  $\Omega$  be the set of all probability measures on  $\mathbb{R}$  such that  $\int f(x)dQ(x) = 0$ .

The classical Gibbs conditioning principle writes as follows:



The limiting distribution  $P^*$  of  $X_1$  conditioned on the family of events  $E_n$  exists and is defined as the unique minimizer of the Kullback-Leibler distance between  $P$  and  $\Omega$ , namely

$$P^* = \arg \min \{K(Q, P), Q \in \Omega\}$$

where

$$K(Q, P) := \int \log \frac{dQ}{dP} dQ$$

whenever  $Q$  is absolutely continuous w.r.t.  $P$ , and  $K(Q, P) = \infty$  otherwise. Also it can be proved that  $P^*$  has a density, which is defined through

$$p^*(x) = \frac{\exp \lambda f(x)}{\Phi_f(\lambda)} p(x)$$

with  $\lambda$  the unique solution of the equation  $m_f(\lambda) = 0$ . Take  $f(x) = x - a$  with  $a$  fixed to obtain

$$p^*(x) = \pi^a(x)$$

with the current notation of this paper.

Consider now the application of the above result to r.v's  $Y_1, \dots, Y_n$  with  $Y_i := (X_i)^2$  where the  $X_i$ 's are i.i.d. and are such that the density of the i.i.d. r.v's  $Y_i$ 's satisfy (2.1) with all the hypotheses stated in this paper. By the Gibbs conditional principle, for *fixed*  $a$ , conditionally on  $(\sum_{i=1}^n Y_i = na)$  the generic r.v.  $Y_1$  has a non degenerate limit distribution

$$p_Y^*(y) := \frac{\exp ty}{E \exp tY_1} p_Y(y)$$

and the limit density of  $X_1$  under  $(\sum_{i=1}^n X_i^2 = na)$  is

$$p_X^*(y) := \frac{\exp tx^2}{E \exp tX_1^2} p_X(y)$$

whereas, when  $a_n \rightarrow \infty$  its limit distribution is degenerate and concentrates around  $a_n$ . As a consequence the distribution of  $X_1$  under the condition  $(\sum_{i=1}^n X_i^2 = na_n)$  concentrates sharply at  $-\sqrt{a_n}$  and  $+\sqrt{a_n}$ .

## 6 EDP under exceedance

The following proposition states the marginally conditional density under condition  $A_n = \{S_n \geq na_n\}$ , we denote this density by  $p_{A_n}$  to differentiate it from  $p_{a_n}$  which is under condition  $\{S_n = na_n\}$ . For the purpose of proof, we need the following lemma, based on Theorem 6.2.1 of Jensen [8], to provide one asymptotic estimation of tail probability  $P(S_n \geq na_n)$  and  $n$ -convolution density  $p(S_n/n = u)$  for  $u > a_n$ .

Define

$$I(x) := xm^{-1}(x) - \log \Phi(m^{-1}(x)). \quad (6.1)$$

**Lemma 6.1.**  $X_1, \dots, X_n$  are i.i.d. random variables with density  $p(x)$  defined in (2.1) and  $h(x) \in \mathfrak{R}$ . Set  $m(t_n) = a_n$ . Suppose when  $n \rightarrow \infty$ , if it holds

$$\frac{\psi(t_n)^2}{\sqrt{n}\psi'(t_n)} \rightarrow 0, \quad (6.2)$$

then it holds

$$P(S_n \geq na_n) = \frac{\exp(-nI(a_n))}{\sqrt{2\pi}\sqrt{nt_ns(t_n)}} \left(1 + o\left(\frac{1}{\sqrt{n}}\right)\right). \quad (6.3)$$

Let further

$$H_n(u) := \frac{\sqrt{n} \exp(-nI(u))}{\sqrt{2\pi}s(t_u)}$$

It then holds

$$\sup_{u > a_n} \frac{p(S_n/n = u)}{H_n(u)} = 1 + o(1/\sqrt{n}). \quad (6.4)$$

Proof: For the density  $p(x)$  defined in (2.1), we show  $g(x)$  is convex when  $x$  is large enough. If  $h(x) \in R_\beta$ , it holds for  $x$  large enough

$$g''(x) = h'(x) = \frac{h(x)}{x}(\beta + \epsilon(x)) > 0. \quad (6.5)$$

If  $h(x) \in R_\infty$ , its reciprocal function  $\psi(x) \in \widetilde{R}_0$ . Set  $x = \psi(u)$ , hence we have for  $x$  large enough

$$g''(x) = h'(x) = \frac{1}{\psi'(u)} = \frac{u}{\psi(u)\epsilon(u)} > 0, \quad (6.6)$$

where the inequality holds since  $\epsilon(u) > 0$  under condition (2.7) when  $u$  is large enough. (6.5) and (6.6) imply that  $g(x)$  is convex for  $x$  large enough.

Therefore, the density  $p(x)$  with  $h(x) \in \mathfrak{R}$  satisfies the conditions of Jensen's Theorem 6.2.1 ([8]). Denote by  $p_n$  the density of  $\bar{X} = (X_1 + \dots + X_n)/n$ . We obtain with the third order's Edgeworth expansion from formula (2.2.6) of ([8])

$$P(S_n \geq na_n) = \frac{\Phi(t_n)^n \exp(-nt_na_n)}{\sqrt{n}t_ns(t_n)} \left( B_0(\lambda_n) + O\left(\frac{\mu_3(t_n)}{6\sqrt{n}s^3(t_n)} B_3(\lambda_n)\right) \right), \quad (6.7)$$

where  $\lambda_n = \sqrt{n}t_ns(t_n)$ ,  $B_0(\lambda_n)$  and  $B_3(\lambda_n)$  are defined by

$$B_0(\lambda_n) = \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{1}{\lambda_n^2} + o\left(\frac{1}{\lambda_n^2}\right) \right), \quad B_3(\lambda_n) \sim -\frac{3}{\sqrt{2\pi}\lambda_n}.$$

We show, under condition (6.2), it holds as  $a_n \rightarrow \infty$

$$\frac{1}{\lambda_n^2} = o\left(\frac{1}{n}\right). \quad (6.8)$$

Since  $n/\lambda_n^2 = 1/(t_n^2 s^2(t_n))$ , (6.8) is equivalent to show

$$t_n^2 s^2(t_n) \rightarrow \infty. \quad (6.9)$$

By Theorem 3.1,  $m(t_n) \sim \psi(t_n)$  and  $s^2(t_n) \sim \psi'(t_n)$ , combined with (??), it holds  $t_n \sim h(a_n)$ .

If  $h \in R_\beta$ , notice that it holds

$$\psi'(t_n) = \frac{1}{h'(\psi(t_n))} = \frac{\psi(t_n)}{h(\psi(t_n))(\beta + \epsilon(\psi(t_n)))} \sim \frac{a_n}{h(a_n)(\beta + \epsilon(\psi(t_n)))},$$

hence we have

$$t_n^2 s^2(t_n) \sim h(a_n)^2 \frac{a_n}{h(a_n)(\beta + \epsilon(\psi(t_n)))} = \frac{a_n h(a_n)}{\beta + \epsilon(\psi(t_n))} \rightarrow \infty. \quad (6.10)$$

If  $h \in R_\infty$ , then  $\psi(t_n) \in \widetilde{R}_0$ , thus it follows

$$t_n^2 s^2(t_n) \sim t_n^2 \frac{\psi(t_n)\epsilon(t_n)}{t_n} = t_n \psi(t_n) \epsilon(t_n) \rightarrow \infty, \quad (6.11)$$

where last step holds from condition (2.7). We have showed (6.8), therefore it holds

$$B_0(\lambda_n) = \frac{1}{\sqrt{2\pi}} \left( 1 + o\left(\frac{1}{n}\right) \right).$$

By (6.9),  $\lambda_n$  goes to  $\infty$  as  $a_n \rightarrow \infty$ , which implies further  $B_3(\lambda_n) \rightarrow 0$ . On the other hand, by (3.1) it holds  $\mu_3/s^3 \rightarrow 0$ . Hence we obtain from (6.7)

$$P(S_n \geq na_n) = \frac{\Phi(t_n)^n \exp(-nt_n a_n)}{\sqrt{2\pi n} t_n s(t_n)} \left( 1 + o\left(\frac{1}{\sqrt{n}}\right) \right),$$

which together with (6.1) gives (6.3).

By Jensen's Theorem 6.2.1 ([8]) and formula (2.2.4) in [8] it follows that

$$p(S_n = na_n) = \frac{\sqrt{n} \Phi(t_n)^n \exp(-nt_n a_n)}{\sqrt{2\pi} s(t_n)} \left( 1 + o\left(\frac{1}{\sqrt{n}}\right) \right),$$

which, together with (6.1), gives (6.4).

**Proposition 6.1.**  $X_1, \dots, X_n$  are i.i.d. random variables with density  $p(x)$  defined in (2.1) and  $h(x) \in \mathfrak{R}$ . Suppose when  $n \rightarrow \infty$ , if it holds

$$\frac{\psi(t_n)^2}{\sqrt{n}\psi'(t_n)} \rightarrow 0, \quad (6.12)$$

and

$$\eta_n \rightarrow 0, \quad \frac{\log n}{nh(a_n)\eta_n} \rightarrow \infty, \quad (6.13)$$

then

$$p_{A_n}(y_1) = p(X_1 = y_1 | S_n \geq na_n) = g_{A_n}(y_1) \left(1 + o\left(\frac{1}{\sqrt{n}}\right)\right),$$

where  $g_{A_n}(y_1) = nt_n s(t_n) e^{nI(a_n)} \int_{a_n}^{a_n + \eta_n} g_\tau(y_1) \exp(-nI(\tau) - \log s(t_\tau)) d\tau$ ,  $g_\tau(y_1)$  is defined as  $g_{a_n}(y_1)$  in Theorem (5.1) on replacing  $a_n$  by  $\tau$ .

Proof: We can denote  $p_{A_n}(y_1)$  by the integration of  $p_{a_n}(y_1)$

$$\begin{aligned} p_{A_n}(y_1) &= \int_{a_n}^{\infty} p(X_1 = y_1 | S_n = n\tau) p(S_n = n\tau | S_n \geq na_n) d\tau \\ &= p(X_1 = y_1) \frac{\int_{a_n}^{\infty} p(S_2^n = n\tau - y_1) d\tau}{p(S_n \geq na_n)} \\ &= \frac{p(X_1 = y_1)}{p(S_n \geq na_n)} P_1 \left(1 + \frac{P_2}{P_1}\right), \end{aligned}$$

where the second equality is obtained by Bayes formula, and  $P_1 = \int_{a_n}^{a_n + \eta_n} p(S_2^n = n\tau - y_1) d\tau$ ,  $P_2 = \int_{a_n + \eta_n}^{\infty} p(S_2^n = n\tau - y_1) d\tau$ ,  $S_2^n = X_2 + \dots + X_n$ . In fact  $P_2$  is one infinitely small term with respect to  $P_1$ , which is proved below. Further we have

$$\begin{aligned} P_2 &= \frac{1}{n} P(S_2^n \geq n(a_n + \eta) - y_1) = \frac{1}{n} P(S_2^n \geq (n-1)c_n), \\ P_1 + P_2 &= \frac{1}{n} P(S_2^n \geq na_n - y_1) = \frac{1}{n} P(S_2^n \geq (n-1)d_n), \end{aligned}$$

where  $c_n = (n(a_n + \eta) - y_1)/(n-1)$  and  $d_n = (na_n - y_1)/(n-1)$ . Denote  $t_{c_n} = m^{-1}(c_n)$  and  $t_{d_n} = m^{-1}(d_n)$ . Using Lemma (6.1), it holds

$$\frac{P_2}{P_1 + P_2} = \left(1 + o\left(\frac{1}{\sqrt{n}}\right)\right) \frac{t_{d_n} s(t_{d_n})}{t_{c_n} s(t_{c_n})} \exp\left(-(n-1)(I(c_n) - I(d_n))\right), \quad (6.14)$$

Using the convexity of the function  $I$ , it holds

$$\begin{aligned}\exp\left(-(n-1)(I(c_n) - I(d_n))\right) &\leq \exp\left(-(n-1)(c_n - d_n)m^{-1}(d_n)\right) \\ &= \exp\left(-n\eta_n m^{-1}(d_n)\right)\end{aligned}$$

Consider  $u \rightarrow m^{-1}(u)$  is increasing, since  $d_n \leq a_n$  as  $a_n \rightarrow \infty$ , it holds  $m^{-1}(d_n) \geq m^{-1}(a_n)$ , hence we get

$$\exp\left(-(n-1)(I(c_n) - I(d_n))\right) \leq \exp\left(-n\eta_n m^{-1}(a_n)\right). \quad (6.15)$$

Using Theorem 3.1, we have  $m^{-1}(a_n) \sim \psi^{-1}(a_n) = h(a_n)$ , thus under condition (6.13) it holds as  $a_n \rightarrow \infty$

$$\exp\left(-(n-1)(I(c_n) - I(d_n))\right) \rightarrow 0.$$

Then we show it holds

$$\frac{t_{d_n}s(t_{d_n})}{t_{c_n}s(t_{c_n})} \rightarrow 1. \quad (6.16)$$

By definition,  $c_n/d_n \rightarrow 1$  as  $a_n \rightarrow \infty$ . if  $h \in R_\beta$ , by (6.10), it holds

$$\left(\frac{t_{d_n}s(t_{d_n})}{t_{c_n}s(t_{c_n})}\right)^2 \sim \left(\frac{d_n h(d_n)}{\beta + \epsilon(\psi(d_n))}\right)^2 \left(\frac{\beta + \epsilon(\psi(c_n))}{c_n h(c_n)}\right)^2 \sim \left(\frac{h(d_n)}{h(c_n)}\right)^2 \rightarrow 1. \quad (6.17)$$

If  $h \in R_\infty$ , notice the function  $t \rightarrow t\psi(t)\epsilon(t)$  is increasing and continuous as  $t$  large enough. By (6.11), it holds

$$t^2 s^2(t) \sim t\psi(t)\epsilon(t), \quad (6.18)$$

consider  $d_n \rightarrow c_n$  as  $n \rightarrow \infty$ , hence we have

$$\left(\frac{t_{d_n}s(t_{d_n})}{t_{c_n}s(t_{c_n})}\right)^2 \sim \frac{d_n\psi(d_n)\epsilon(d_n)}{c_n\psi(c_n)\epsilon(c_n)} \rightarrow 1. \quad (6.19)$$

Using (6.14), (6.15) and (6.16), we obtain

$$\frac{P_2}{P_1 + P_2} \leq 2 \exp\left(-nm^{-1}(a_n)\eta_n\right),$$

which, together with condition (6.13), it holds

$$\frac{P_2}{P_1} = o\left(\frac{1}{\sqrt{n}}\right).$$

Therefore we can approximate  $p_{A_n}(y_1)$  by

$$p_{A_n}(y_1) = \left(1 + o\left(\frac{1}{\sqrt{n}}\right)\right) \int_{a_n}^{a_n + \eta_n} p(X_1 = y_1 | S_n = n\tau) p(S_n = n\tau | S_n \geq na_n) d\tau. \quad (6.20)$$

According to Lemma 6.1, it follows when  $\tau \in [a_n, a_n + \eta_n]$

$$p(S_n = n\tau | S_n \geq na_n) = \left(1 + o\left(\frac{1}{\sqrt{n}}\right)\right) \frac{nm^{-1}(a_n)s(t_n)}{s(t_\tau)} \exp(-n(I(\tau) - I(a_n))), \quad (6.21)$$

where  $m(t_n) = a_n$ ,  $m(t_\tau) = \tau$ . Inserting (6.20) into (6.21), we obtain

$$p_{A_n}(y_1) = \left(1 + o\left(\frac{1}{\sqrt{n}}\right)\right) nt_n s(t_n) e^{nI(a_n)} \int_{a_n}^{a_n + \eta_n} g_\tau(y_1) \exp(-nI(\tau) - \log s(t_\tau)) d\tau,$$

this completes the proof.

## 7 Appendix

For density functions  $p(x)$  defined in (2.1) satisfying also  $h(x) \in \mathfrak{R}$ , denote by  $\psi(x)$  the reciprocal function of  $h(x)$  and  $\sigma^2(v) = (h'(v))^{-1}$ ,  $v \in \mathbb{R}_+$ . For brevity, we write  $\hat{x}, \sigma, l$  instead of  $\hat{x}(t), \sigma(\psi(t)), l(t)$ .

When  $t$  is given,  $K(x, t)$  attain its maximum at  $\hat{x} = \psi(t)$ . The fourth order Taylor expansion of  $K(x, t)$  on  $x \in [\hat{x} - \sigma l, \hat{x} + \sigma l]$  yields

$$K(x, t) = K(\hat{x}, t) - \frac{1}{2}h'(\hat{x})(x - \hat{x})^2 - \frac{1}{6}h''(\hat{x})(x - \hat{x})^3 + \epsilon(x, t), \quad (7.1)$$

with some  $\theta \in [0, 1]$

$$\epsilon(x, t) = -\frac{1}{24}h'''(\hat{x} + \theta(x - \hat{x}))(x - \hat{x})^4. \quad (7.2)$$

**Lemma 7.1.** *For  $p(x)$  in (2.1),  $h(x) \in \mathfrak{R}$ , it holds when  $t \rightarrow \infty$ ,*

$$\frac{|\log \sigma(\psi(t))|}{\int_1^t \psi(u) du} \longrightarrow 0. \quad (7.3)$$

Proof: If  $h(x) \in R_\beta$ , by Theorem (1.5.12) of [1], there exists some slowly varying function such that it holds  $\psi(x) \sim x^{1/\beta} l_1(x)$ . Hence it holds as  $t \rightarrow \infty$  (see [7], Chapter 8)

$$\int_1^t \psi(u) du \sim t^{1+\frac{1}{\alpha}} l_1(t). \quad (7.4)$$

On the other hand,  $h'(x) = x^{\beta-1}l(x)(\beta + \epsilon(x))$ , thus we have as  $x \rightarrow \infty$

$$\begin{aligned} |\log \sigma(x)| &= \left| \log (h'(x))^{-\frac{1}{2}} \right| = \left| \frac{1}{2}((\beta - 1) \log x + \log l(x) + \log(\beta + \epsilon(x))) \right| \\ &\leq \frac{1}{2}(\beta + 1) \log x, \end{aligned}$$

set  $x = \psi(t)$ , then when  $t \rightarrow \infty$ , it holds  $x < 2t^{1/\beta}l_1(t) < t^{1/\beta+1}$ , hence we have

$$|\log \sigma(\psi(t))| < \frac{(\beta + 1)^2}{2\beta} \log t,$$

which, together with (7.4), yields (7.22).

If  $h(x) \in R_\infty$ , then by definition  $\psi(x) \in \widetilde{R}_0$  is slowly varying as  $x \rightarrow \infty$ . Hence it holds as  $t \rightarrow \infty$  (see [7], Chapter 8)

$$\int_1^t \psi(u) du \sim t\psi(t). \quad (7.5)$$

And now we have  $h'(x) = 1/\psi'(v)$  with  $x = \psi(v)$ . Therefore it follows

$$|\log \sigma(x)| = \left| \log (h'(x))^{-\frac{1}{2}} \right| = \frac{1}{2} |\log \psi'(v)|,$$

Set  $x = \psi(t)$ , then  $v = t$ , consider  $\psi(t) \in \widetilde{R}_0$ , thus we have

$$\begin{aligned} |\log \sigma(\psi(t))| &= \frac{1}{2} |\log \psi'(t)| = \frac{1}{2} \left| \log \left( \psi(t) \frac{\epsilon(t)}{t} \right) \right| \\ &= \frac{1}{2} |\log \psi(t) + \log \epsilon(t) - \log t| \\ &\leq \log t + \frac{1}{2} |\log \epsilon(t)| \leq 2 \log t, \end{aligned} \quad (7.6)$$

where last inequality follows from (2.6). (7.5) and (7.6) imply (7.22). This completes the proof.

**Lemma 7.2.** *For  $p(x)$  in (2.1),  $h \in \mathfrak{R}$ , then for any varying slowly function  $l(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , it holds*

$$\sup_{|x| \leq \sigma l} h'''(\hat{x} + x) \sigma^4 l^4 \longrightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (7.7)$$

**Proof: Case 1:**  $h \in R_\beta$ . We have  $h(x) = x^\beta l_0(x)$ ,  $l_0(x) \in R_0$ ,  $\beta > 0$ . Hence it holds

$$h''(x) = \beta(\beta - 1)x^{\beta-2}l_0(x) + 2\beta x^{\beta-1}l_0'(x) + x^\beta l_0''(x). \quad (7.8)$$

and

$$h'''(x) = \beta(\beta-1)(\beta-2)x^{\beta-3}l_0(x) + 3\beta(\beta-1)x^{\beta-2}l_0'(x) + 3\beta x^{\beta-1}l_0''(x) + x^\beta l_0'''(x). \quad (7.9)$$

Consider  $l(x) \in R_0$ , it is easy to obtain

$$l_0'(x) = \frac{l_0(x)}{x}\epsilon(x), \quad l_0''(x) = \frac{l_0(x)}{x^2}(\epsilon^2(x) + x\epsilon'(x) - \epsilon(x)), \quad (7.10)$$

and

$$l_0'''(x) = \frac{l_0(x)}{x^3}(\epsilon^3(x) + 3x\epsilon'(x)\epsilon(x) - 3\epsilon^2(x) - 2x\epsilon''(x) + 2\epsilon(x) + x^2\epsilon'''(x)).$$

Under condition (2.5), there exists some positive constant  $Q$  such that it holds

$$|l_0''(x)| \leq Q \frac{l_0(x)}{x^2}, \quad |l_0'''(x)| \leq Q \frac{l_0(x)}{x^3},$$

which, together with (7.9), yields with some positive constant  $Q_1$

$$|h'''(x)| \leq Q_1 \frac{h(x)}{x^3}. \quad (7.11)$$

By definition, we have  $\sigma^2(x) = 1/h'(x) = x/(h(x)(\beta + \epsilon(x)))$ , thus it follows

$$\sigma^2 = \sigma^2(\hat{x}) = \frac{\hat{x}}{h(\hat{x})(\beta + \epsilon(\hat{x}))} = \frac{\psi(t)}{t(\beta + \epsilon(\psi(t)))} = \frac{\psi(t)}{\beta t}(1 + o(1)), \quad (7.12)$$

this implies  $\sigma l = o(\psi(t)) = o(\hat{x})$ . Thus we get with (7.11)

$$\sup_{|x| \leq \sigma l} |h'''(\hat{x} + x)| \leq \sup_{|x| \leq \sigma l} Q_1 \frac{h(\hat{x} + x)}{(\hat{x} + x)^3} \leq Q_2 \frac{t}{\psi^3(t)}, \quad (7.13)$$

where  $Q_2$  is some positive constant. Combined with (7.12), we obtain

$$\sup_{|x| \leq \sigma l} |h'''(\hat{x} + x)| \sigma^4 l^4 \leq Q_2 \frac{t}{\psi^3(t)} \sigma^4 l^4 = \frac{Q_2 l^4}{\beta^2 t \psi(t)} \longrightarrow 0,$$

as sought.

**Case 2:**  $h \in R_\infty$ . Since  $\hat{x} = \psi(t)$ , we have  $h(\hat{x}) = t$ . Thus it holds

$$h'(\hat{x}) = \frac{1}{\psi'(t)} \quad \text{and} \quad h''(\hat{x}) = -\frac{\psi''(t)}{(\psi'(t))^3}, \quad (7.14)$$



further we get

$$h'''(\hat{x}) = -\frac{\psi'''(t)\psi'(t) - 3(\psi''(t))^2}{(\psi'(t))^4}. \quad (7.15)$$

Notice if  $h(\hat{x}) \in R_\infty$ , then  $\psi(t) \in \widetilde{R_0}$ . Therefore we obtain

$$\psi'(t) = \frac{\psi(t)}{t}\epsilon(t), \quad (7.16)$$

and

$$\begin{aligned} \psi''(t) &= -\frac{\psi(t)}{t^2}\epsilon(t)\left(1 - \epsilon(t) - \frac{t\epsilon'(t)}{\epsilon(t)}\right) \\ &= -\frac{\psi(t)}{t^2}\epsilon(t)(1 + o(1)) \quad \text{as } t \rightarrow \infty, \end{aligned} \quad (7.17)$$

where last equality holds from (2.6). Using (2.6) once again, we have also  $\psi'''(t)$

$$\begin{aligned} \psi'''(t) &= \frac{\psi(t)}{t^3}\epsilon(t)\left(2 + \epsilon^2(t) + 3t\epsilon'(t) - 3\epsilon(t) - \frac{2t\epsilon'(t)}{\epsilon(t)} + \frac{t^2\epsilon''(t)}{\epsilon(t)}\right) \\ &= \frac{\psi(t)}{t^3}\epsilon(t)(2 + o(1)) \quad \text{as } t \rightarrow \infty. \end{aligned} \quad (7.18)$$

Put (7.16) (7.17) and (7.18) into (7.15) we get

$$h'''(\hat{x}) = \frac{1}{\psi^2(t)\epsilon^2(t)}(1 + o(1))$$

Thus by (2.7) it holds as  $t \rightarrow \infty$

$$\begin{aligned} \sup_{|v| \leq t/4} h'''(\psi(t+v)) &= \sup_{|v| \leq t/4} \frac{1}{\psi^2(t+v)\epsilon^2(t+v)}(1 + o(1)) \\ &\leq \sup_{|v| \leq t/4} \frac{2\sqrt{t+v}}{\psi^2(t+v)} \leq \frac{3\sqrt{t}}{\psi^2(t)}, \end{aligned} \quad (7.19)$$

where last inequality holds from the slowly varying propriety:  $\psi(t+v) \sim \psi(t)$ . Using  $\sigma = (h'(\hat{x}))^{-1/2}$ , it holds

$$\sup_{|v| \leq t/4} h'''(\psi(t+v))\sigma^4 \leq \frac{3\sqrt{t}}{\psi^2(t)} \frac{1}{(h'(\hat{x}))^2} = \frac{3\sqrt{t}}{\psi^2(t)} \frac{\psi^2(t)\epsilon^2(t)}{t^2} = \frac{3\epsilon^2(t)}{t^{3/2}} \rightarrow 0,$$

where  $\epsilon(t) \rightarrow 0$  and  $\psi(t)$  varies slowly. Hence for any slowly varying function  $l(t) \rightarrow \infty$  it holds as  $t \rightarrow \infty$

$$\sup_{|v| \leq t/4} h'''(\psi(t+v)) \sigma^4 l^4 \rightarrow 0.$$

Consider  $\psi(t) \in \widetilde{R}_0$ , thus  $\psi(t)$  is increasing, we have the relation

$$\sup_{|v| \leq t/4} h'''(\psi(t+v)) = \sup_{|\zeta| \leq [\zeta_1, \zeta_2]} h'''(\hat{x} + \zeta),$$

where

$$\zeta_1 = \psi(3t/4) - \hat{x}, \quad \zeta_2 = \psi(5t/4) - \hat{x}.$$

Hence we have showed

$$\sup_{|\zeta| \leq [\zeta_1, \zeta_2]} h'''(\hat{x} + \zeta) \sigma^4 l^4 \rightarrow 0.$$

For completing the proof, it remains to show

$$\sigma l \leq \min(|\zeta_1|, \zeta_2) \quad \text{as } t \rightarrow \infty. \quad (7.20)$$

Perform first order Taylor expansion of  $\psi(3t/4)$  at  $t$ , for some  $\alpha \in [0, 1]$ , it holds

$$\zeta_1 = \psi(3t/4) - \hat{x} = \psi(3t/4) - \psi(t) = -\psi'(t - \alpha t/4) \frac{t}{4} = -\frac{\psi(t - \alpha t/4)}{4 - \alpha} \epsilon(t - \alpha t/4),$$

thus using (2.7) and slowly varying propriety of  $\psi(t)$  we get as  $t \rightarrow \infty$

$$|\zeta_1| \geq \frac{\psi(t - \alpha t/4)}{4} \epsilon(t - \alpha t/4) \geq \frac{\psi(t)}{5} \epsilon(t - \alpha t/4) \geq \frac{\psi(t)}{5t^{1/4}}. \quad (7.21)$$

On the other hand, we have  $\sigma = (h'(\hat{x}))^{-1/2} = (\psi(t)\epsilon(t)/t)^{1/2}$ , which, together with (7.21), yields

$$\frac{\sigma}{|\zeta_1|} \leq 5 \sqrt{\frac{\epsilon(t)}{\psi(t)\sqrt{t}}} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

which implies for any slowly varying function  $l(t)$  it holds  $\sigma l = o(|\zeta_1|)$ . By the same way, it is easy to show  $\sigma l = o(\zeta_2)$ . Hence (7.20) holds, as sought.

**Lemma 7.3.** For  $p(x)$  in (2.1),  $h \in \mathfrak{R}$ , then for any varying slowly function  $l(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , it holds

$$\sup_{|x| \leq \sigma l} \frac{h'''(\hat{x} + x)}{h''(\hat{x})} \sigma l \longrightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (7.22)$$

and

$$h''(\hat{x}) \sigma^3 l \longrightarrow 0. \quad (7.23)$$

Proof: **Case 1:** Using (7.8) and (7.10), we get  $h''(x) = (\beta(\beta - 1) + o(1))x^{\beta-2}l_0(x)$  as  $x \rightarrow \infty$ , where  $l_0(x) \in R_0$ . Hence it holds

$$h''(\hat{x}) = (\beta(\beta - 1) + o(1))\psi(t)^{\beta-2}l_0(\psi(t)), \quad (7.24)$$

which, together with (7.12) and (7.13), yields with some positive constant  $Q_3$

$$\sup_{|x| \leq \sigma l} \left| \frac{h'''(\hat{x} + x)}{h''(\hat{x})} \sigma l \right| \leq Q_3 \frac{t}{\psi^3(t)} \frac{1}{\psi(t)^{\beta-2}l_0(\psi(t))} \sqrt{\frac{\psi(t)}{\beta t}} l = \frac{Q_3}{\sqrt{\beta}} \frac{\sqrt{t}}{\psi(t)^{\beta+1/2}l_0(\psi(t))} l.$$

Notice  $\psi(t) \sim t^{1/\beta}l_1(t)$  for some slowly varying function  $l_1(t)$ , then it holds  $\sqrt{t}l = o(\psi(t)^{\beta+1/2})$ . Hence we get (7.22).

From (7.12) and (7.24), we obtain as  $t \rightarrow \infty$

$$\begin{aligned} h''(\hat{x}) \sigma^3 l &= (\beta(\beta - 1) + o(1))\psi(t)^{\beta-2}l_0(\psi(t)) \left( \frac{\psi(t)}{\beta t} \right)^{3/2} l \\ &= (\beta(\beta - 1) + o(1)) \frac{\psi(t)^{\beta-1/2}}{\beta^{3/2}t^{3/2}} l_0(\psi(t)) l \leq \frac{1}{\sqrt{t}}, \end{aligned} \quad (7.25)$$

where last inequality holds since  $\psi(t)^{\beta-1/2}/t^{3/2} \sim l_1(t)^{\beta-1/2}/t^{1/2+1/2\beta}$  as  $t \rightarrow \infty$ . This implies (7.23) holds.

**Case 2:** Using (7.14) and (7.17) we obtain

$$h''(\hat{x}) = -\frac{\psi''(t)}{(\psi'(t))^3} = \frac{t}{\psi^2(t)\epsilon^2(t)}(1 + o(1)). \quad (7.26)$$

Combine (7.19) and (7.26), using  $\sigma = (h'(\hat{x}))^{-1/2}$ , we have as  $t \rightarrow \infty$

$$\sup_{|v| \leq t/4} \frac{h'''(\psi(t+v))}{h''(\hat{x})} \sigma \leq \frac{4\epsilon^2(t)}{\sqrt{t}} \frac{1}{\sqrt{h'(\hat{x})}} = \frac{4\epsilon(t)^{5/2}\sqrt{\psi(t)}}{t} \rightarrow 0,$$

where  $\epsilon(t) \rightarrow 0$  and  $\psi(t)$  varies slowly. Hence for arbitrarily slowly varying function  $l(t)$  it holds as  $t \rightarrow \infty$

$$\sup_{|v| \leq t/4} \frac{h'''(\psi(t+v))}{h''(\hat{x})} \sigma l \rightarrow 0.$$

Define  $\zeta_1, \zeta_2$  as in Lemma 7.2, we have showed

$$\sup_{|\zeta| \leq [\zeta_1, \zeta_2]} \frac{h'''(\hat{x} + \zeta)}{h''(\hat{x})} \sigma l \rightarrow 0.$$

(7.22) is obtained by using (7.20). Using (7.26), for any slowly varying function, it holds

$$h''(\hat{x}) \sigma^3 l = \frac{l}{\sqrt{\psi(t)\epsilon(t)t}} \rightarrow 0.$$

Hence the proof.

**Lemma 7.4.** For  $p(x)$  in (2.1),  $h \in \mathfrak{R}$ , then for any slowly varying function  $l(t) \rightarrow \infty$  as  $t \rightarrow \infty$  such that it holds

$$\sup_{y \in [-l, l]} \frac{|\xi(\sigma y + \hat{x}, t)|}{h''(\hat{x}) \sigma^3} \rightarrow 0,$$

where  $\xi(x, t) = \epsilon(x, t) + q(x)$ .

Proof: For  $y \in [-l, l]$ , by (7.2) and Lemma 7.3 it holds as  $t \rightarrow \infty$

$$\frac{|\epsilon(\sigma y + \hat{x}, t)|}{h''(\hat{x}) \sigma^3} \leq \sup_{|x| \leq \sigma l} \left| \frac{h'''(\hat{x} + x)}{h''(\hat{x})} \right| \sigma l \rightarrow 0. \quad (7.27)$$

Under condition (2.2), set  $x = \psi(t)$ , we get

$$\sup_{|v - \psi(t)| \leq \vartheta \psi(t)} |q(v)| \leq \frac{1}{\sqrt{t\psi(t)}},$$

and it holds for any slowly varying function  $l(t)$  as  $t \rightarrow \infty$

$$\frac{\sigma l}{\vartheta \psi(t)} = \frac{\sqrt{\psi'(t)} l}{\vartheta \psi(t)} = \sqrt{\frac{\epsilon(t)}{t\psi(t)}} \frac{l}{\vartheta} \rightarrow 0,$$

hence we obtain

$$\sup_{|v - \psi(t)| \leq \sigma l} |q(v)| \leq \frac{1}{\sqrt{t\psi(t)}}.$$

Using this inequality and (7.26), when  $y \in [-l, l]$ , it holds as  $t \rightarrow \infty$

$$\frac{|q(\sigma y + \hat{x})|}{h''(\hat{x}) \sigma^3} = |q(\sigma y + \hat{x})| \sqrt{\psi(t)\epsilon(t)t} \leq \sup_{|v - \psi(t)| \leq \sigma l} |q(v)| \sqrt{\psi(t)\epsilon(t)t} \leq \sqrt{\epsilon(t)} \rightarrow 0,$$

which, together with (7.27), completes the proof.

**Lemma 7.5.** For  $p(x)$  belonging to (2.1),  $h(x) \in \mathfrak{R}$ ,  $\alpha \in \mathbb{N}$ , denote by

$$\Psi(t, \alpha) := \int_0^\infty (x - \hat{x})^\alpha e^{tx} p(x) dx,$$

then there exists some slowly varying function  $l(t)$  such that it holds as  $t \rightarrow \infty$

$$\Psi(t, \alpha) = c\sigma^{\alpha+1} e^{K(\hat{x}, t)} T_1(t, \alpha) (1 + o(1)),$$

where

$$T_1(t, \alpha) = \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^\alpha \exp\left(-\frac{y^2}{2}\right) dy - \frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^{3+\alpha} \exp\left(-\frac{y^2}{2}\right) dy.$$

Proof: By Lemma 7.2, for any slowly varying function  $l(t)$  it holds as  $t \rightarrow \infty$

$$\sup_{|x-\hat{x}| \leq \sigma l} |\epsilon(x, t)| \rightarrow 0.$$

Given a slowly varying function  $l$  with  $l(t) \rightarrow \infty$  and define the interval  $I_t$  as follows

$$I_t := \left( -\frac{l^{1/3}\sigma}{\sqrt{2}}, \frac{l^{1/3}\sigma}{\sqrt{2}} \right).$$

For large enough  $\tau$ , when  $t \rightarrow \infty$  we can partition  $\mathbb{R}_+$  as

$$\mathbb{R}_+ = \{x : 0 < x < \tau\} \cup \{x : x \in \hat{x} + I_t\} \cup \{x : x \geq \tau, x \notin \hat{x} + I_t\},$$

where  $\tau$  large enough such that it holds for  $x > \tau$

$$p(x) < 2ce^{-g(x)}. \quad (7.28)$$

Obviously, for fixed  $\tau$ ,  $\{x : 0 < x < \tau\} \cap \{x : x \in \hat{x} + I_t\} = \emptyset$  since for large  $t$  we have  $\min(x : x \in \hat{x} + I_t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Hence it holds

$$\begin{aligned} \Psi(t, \alpha) &= \int_0^\tau (x - \hat{x})^\alpha e^{tx} p(x) dx + \int_{x \in \hat{x} + I_t} (x - \hat{x})^\alpha e^{tx} p(x) dx + \int_{x \notin \hat{x} + I_t, x > \tau} (x - \hat{x})^\alpha e^{tx} p(x) dx \\ &:= \Psi_1(t, \alpha) + \Psi_2(t, \alpha) + \Psi_3(t, \alpha). \end{aligned} \quad (7.29)$$

We estimate sequentially  $\Psi_1(t, \alpha)$ ,  $\Psi_2(t, \alpha)$ ,  $\Psi_3(t, \alpha)$  in **Step 1**, **Step 2** and **Step 3**.

**Step 1:** Using (7.28), for  $\tau$  large enough, we have

$$\begin{aligned} |\Psi_1(t, \alpha)| &\leq \int_0^\tau |x - \hat{x}|^\alpha e^{tx} p(x) dx \leq 2c \int_0^\tau |x - \hat{x}|^\alpha e^{tx-g(x)} dx \\ &\leq 2c \int_0^\tau \hat{x}^\alpha e^{tx} dx \leq 2ct^{-1} \hat{x}^\alpha e^{t\tau}. \end{aligned} \quad (7.30)$$

We show it holds for  $h \in \mathfrak{R}$  as  $t \rightarrow \infty$

$$t^{-1}\hat{x}^\alpha e^{t\tau} = o(\sigma^{\alpha+1} e^{K(\hat{x},t)} h''(\hat{x}) \sigma^3). \quad (7.31)$$

(7.31) is equivalent to

$$\sigma^{-\alpha-4} t^{-1} \hat{x}^\alpha e^{t\tau} (h''(\hat{x}))^{-1} = o(e^{K(\hat{x},t)}),$$

which is implied by

$$\exp \left( -(\alpha + 4) \log \sigma - \log t + \alpha \log \hat{x} + \tau t - \log h''(\hat{x}) \right) = o(e^{K(\hat{x},t)}).$$

By Lemma (7.1), we know  $\log \sigma = o(e^{K(\hat{x},t)})$  as  $t \rightarrow \infty$ . So it remains to show  $t = o(e^{K(\hat{x},t)})$ ,  $\log \hat{x} = o(e^{K(\hat{x},t)})$  and  $\log h''(\hat{x}) = o(e^{K(\hat{x},t)})$ . Since  $\hat{x} = \psi(t)$ , it holds

$$K(\hat{x}, t) = t\psi(t) - g(\psi(t)) = \int_1^t \psi(u) du + \psi(1) - g(1), \quad (7.32)$$

where the second equality can be easily verified by the change of variable  $u = h(v)$ .

If  $h(x) \in R_\beta$ , by Theorem (1.5.12) of [1], it holds  $\psi(x) \sim x^{1/\beta} l_1(x)$  with some slowly varying function  $l_1(x)$ . (7.4) and (7.32) yield  $t = o(e^{K(\hat{x},t)})$ . In addition,  $\log \hat{x} = \log \psi(t) \sim (1/\beta) \log t = o(e^{K(\hat{x},t)})$ . By (7.24), it holds  $\log h''(\hat{x}) = o(t)$ . Thus (7.31) holds.

If  $h(x) \in R_\infty$ ,  $\psi(x) \in \widetilde{R}_0$  is slowly varying as  $x \rightarrow \infty$ . Therefore, by (7.5) and (7.32), it holds  $t = o(e^{K(\hat{x},t)})$  and  $\log \hat{x} = \log \psi(t) = o(e^{K(\hat{x},t)})$ . Using (7.26), we have  $\log h''(\hat{x}) \sim \log t - 2 \log \hat{x} - 2 \log \epsilon(t)$ . Under condition (2.7),  $\log \epsilon(t) = o(t)$ , thus it holds  $\log h''(\hat{x}) = o(t)$ . We get (7.31).

(7.30) and (7.31) yield together

$$|\Psi_1(t, \alpha)| = o(\sigma^{\alpha+1} e^{K(\hat{x},t)} h''(\hat{x}) \sigma^3). \quad (7.33)$$

**Step 2:** Notice  $\min(x : x \in \hat{x} + I_t) \rightarrow \infty$  as  $t \rightarrow \infty$ , which implies both  $\epsilon(x, t)$  and  $q(x)$  go to 0 when  $x \in \hat{x} + I_t$ . Using (2.1) and (7.1), then it holds as  $t \rightarrow \infty$

$$\begin{aligned} \Psi_2(t, \alpha) &= \int_{x \in \hat{x} + I_t} (x - \hat{x})^\alpha c \exp(K(x, t) + q(x)) dx \\ &= \int_{x \in \hat{x} + I_t} (x - \hat{x})^\alpha c \exp \left( K(\hat{x}, t) - \frac{1}{2} h'(\hat{x}) (x - \hat{x})^2 \right. \\ &\quad \left. - \frac{1}{6} h''(\hat{x}) (x - \hat{x})^3 + \xi(x, t) \right) dx, \end{aligned}$$

where  $\xi(x, t) = \epsilon(x, t) + q(x)$ . Make the change of variable  $y = (x - \hat{x})/\sigma$ , it holds

$$\Psi_2(t, \alpha) = c \sigma^{\alpha+1} \exp(K(\hat{x}, t)) \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^\alpha \exp \left( -\frac{y^2}{2} - \frac{h''(\hat{x}) \sigma^3}{6} y^3 + \xi(\sigma y + \hat{x}, t) \right) dy. \quad (7.34)$$

On  $y \in (-l^{1/3}/\sqrt{2}, l^{1/3}/\sqrt{2})$ , by (7.23),  $|h''(\hat{x})\sigma^3 y^3| \leq |h''(\hat{x})\sigma^3 l| \rightarrow 0$  as  $t \rightarrow \infty$ . Perform the first order Taylor expansion, it holds as  $t \rightarrow \infty$

$$\exp\left(-\frac{h''(\hat{x})\sigma^3}{6}y^3 + \xi(\sigma y + \hat{x}, t)\right) = 1 - \frac{h''(\hat{x})\sigma^3}{6}y^3 + \xi(\sigma y + \hat{x}, t) + o_1(t, y),$$

where

$$o_1(t, y) = o\left(-\frac{h''(\hat{x})\sigma^3}{6}y^3 + \xi(\sigma y + \hat{x}, t)\right).$$

Hence we obtain

$$\begin{aligned} & \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^\alpha \exp\left(-\frac{y^2}{2} - \frac{h''(\hat{x})\sigma^3}{6}y^3 + \xi(\sigma y + \hat{x}, t)\right) dy \\ &= \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} \left(1 - \frac{h''(\hat{x})\sigma^3}{6}y^3 + \xi(\sigma y + \hat{x}, t) + o_1(t, y)\right) y^\alpha \exp\left(-\frac{y^2}{2}\right) dy \\ &= \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^\alpha \exp\left(-\frac{y^2}{2}\right) dy - \frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^{3+\alpha} \exp\left(-\frac{y^2}{2}\right) dy \\ & \quad + \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} \left(\xi(\sigma y + \hat{x}, t) + o_1(t, y)\right) y^\alpha \exp\left(-\frac{y^2}{2}\right) dy. \end{aligned}$$

Define  $T_1(t, \alpha)$  and  $T_2(t, \alpha)$  as follows

$$\begin{aligned} T_1(t, \alpha) &= \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^\alpha \exp\left(-\frac{y^2}{2}\right) dy - \frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^{3+\alpha} \exp\left(-\frac{y^2}{2}\right) dy, \\ T_2(t, \alpha) &= \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} \left(\xi(\sigma y + \hat{x}, t) + o_1(t, y)\right) y^\alpha \exp\left(-\frac{y^2}{2}\right) dy. \end{aligned} \tag{7.35}$$

As for  $T_2(t, \alpha)$ , it holds

$$\begin{aligned}
|T_2(t, \alpha)| &\leq \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} \left( |\xi(\sigma y + \hat{x}, t)| + |o_1(t, y)| \right) |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy \\
&\leq \sup_{y \in [-l, l]} |\xi(\sigma y + \hat{x}, t)| \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy + \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} |o_1(t, y)| |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy \\
&\leq \sup_{y \in [-l, l]} |\xi(\sigma y + \hat{x}, t)| \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy \\
&\quad + \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} \left( \left| o\left(\frac{h''(\hat{x})\sigma^3}{6} y^3\right) \right| + |o(\xi(\sigma y + \hat{x}, t))| \right) |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy \\
&\leq 2 \sup_{y \in [-l, l]} |\xi(\sigma y + \hat{x}, t)| \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy + |o(h''(\hat{x})\sigma^3)| \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} |y|^{3+\alpha} \exp\left(-\frac{y^2}{2}\right) dy \\
&= |o(h''(\hat{x})\sigma^3)| \left( \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy + \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} |y|^{3+\alpha} \exp\left(-\frac{y^2}{2}\right) dy \right),
\end{aligned}$$

where last equality holds from Lemma 7.4. Since the integrals in the last equality are both bounded, it holds as  $t \rightarrow \infty$

$$T_2(t, \alpha) = o(h''(\hat{x})\sigma^3).$$

When  $\alpha$  is even, the second term of  $T_1(t, \alpha)$  vanishes. When  $\alpha$  is odd, the first term of  $T_1(t, \alpha)$  vanishes. Obviously,  $T_1(t, \alpha)$  is at least the same order than  $h''(\hat{x})\sigma^3$ . Therefore it follows as  $t \rightarrow \infty$

$$T_2(t, \alpha) = o(T_1(t, \alpha)). \quad (7.36)$$

Using (7.34), (7.35) and (7.36) we get

$$\Psi_2(t, \alpha) = c\sigma^{\alpha+1} \exp(K(\hat{x}, t)) T_1(t, \alpha) (1 + o(1)). \quad (7.37)$$

**Step 3:** Given  $h \in \mathfrak{R}$ , for any  $t$ ,  $K(x, t)$  as a function of  $x$  ( $x > \tau$ ) is concave since

$$K''(x, t) = -h'(x) < 0.$$

Thus we get for  $x \notin \hat{x} + I_t$  and  $x > \tau$

$$K(x, t) - K(\hat{x}, t) \leq \frac{K(\hat{x} + \frac{l^{1/3}\sigma}{\sqrt{2}} \operatorname{sgn}(x - \hat{x}), t) - K(\hat{x}, t)}{\frac{l^{1/3}\sigma}{\sqrt{2}} \operatorname{sgn}(x - \hat{x})} (x - \hat{x}), \quad (7.38)$$



where

$$\operatorname{sgn}(x - \hat{x}) = \begin{cases} 1 & \text{if } x \geq \hat{x}, \\ -1 & \text{if } x < \hat{x}. \end{cases}$$

Using (7.1), we get

$$K(\hat{x} + \frac{l^{1/3}\sigma}{\sqrt{2}}\operatorname{sgn}(x - \hat{x}), t) - K(\hat{x}, t) \leq -\frac{1}{8}h'(\hat{x})l^{2/3}\sigma^2 = -\frac{1}{8}l^{2/3},$$

which, combined with (7.38), yields

$$K(x, t) - K(\hat{x}, t) \leq -\frac{\sqrt{2}}{8}l^{1/3}\sigma^{-1}|x - \hat{x}|.$$

We obtain

$$\begin{aligned} |\Psi_3(t, \alpha)| &\leq 2c \int_{x \notin \hat{x} + I_t, x > \tau} |x - \hat{x}|^\alpha \exp(K(x, t)) dx \\ &\leq 2c \int_{|x - \hat{x}| > \frac{l^{1/3}\sigma}{\sqrt{2}}} |x - \hat{x}|^\alpha \exp(K(x, t)) dx \\ &\leq 2ce^{K(\hat{x}, t)} \int_{|x - \hat{x}| > \frac{l^{1/3}\sigma}{\sqrt{2}}} |x - \hat{x}|^\alpha \exp\left(-\frac{\sqrt{2}}{8}l^{1/3}\sigma^{-1}|x - \hat{x}|\right) dx \\ &= 2ce^{K(\hat{x}, t)}\sigma^{\alpha+1} \int_{|y| > \frac{l^{1/3}}{\sqrt{2}}} |y|^\alpha \exp\left(-\frac{\sqrt{2}}{8}l^{1/3}|y|\right) dy \\ &= 2ce^{K(\hat{x}, t)}\sigma^{\alpha+1} \int_{|y| > \frac{l^{1/3}}{\sqrt{2}}} \exp\left(-\frac{\sqrt{2}}{8}l^{1/3}|y| + \alpha \log |y|\right) dy \\ &= 2ce^{K(\hat{x}, t)}\sigma^{\alpha+1} \left(2e^{-l^{2/3}/8} (1 + o(1))\right), \end{aligned}$$

where last equality holds when  $l \rightarrow \infty$  (see e.g. Theorem 4.12.10 of [1]). With (7.37), we obtain

$$\left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq \frac{8e^{-l^{2/3}/8}}{|T_1(t, \alpha)|}.$$

In **Step 2**, we know  $T_1(t, \alpha)$  has at least the order  $h''(\hat{x})\sigma^3$ . Hence there exists some positive constant  $Q$  and  $l_2(t) \rightarrow \infty$  such that it holds as  $t \rightarrow \infty$

$$\left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq \frac{Qe^{-l_2^{2/3}/8}}{h''(\hat{x})\sigma^3}.$$

For example, we can take  $l_2(t) = (\log t)^3$ .

If  $h \in R_\beta$ , by (7.25), it is easy to know  $h''(\hat{x})\sigma^3 \geq 1/t^{1+1/(2\beta)}$ , thus we have

$$\left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq Q \exp \left( -l_2^{2/3}/8 + (1 + 1/(2\beta)) \log t \right) \longrightarrow 0.$$

If  $h \in R_\infty$ , using (7.26), then it holds as  $t \rightarrow \infty$

$$\begin{aligned} \left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| &\leq 2Q \exp \left( -l_2^{2/3}/8 + \log \sqrt{t\psi(t)\epsilon(t)} \right) \\ &= 2Q \exp \left( -l_2^{2/3}/8 + (1/2)(\log t + \log \psi(t) + \log \epsilon(t)) \right) \\ &\longrightarrow 0, \end{aligned} \tag{7.39}$$

where last line holds since  $\log \psi(t) = O(\log t)$ . The proof is completed by combining (7.29), (7.33), (7.37) and (7.39).

**Proof of Theorem 3.1:** By Lemma 7.5, if  $\alpha = 0$ , it holds  $T_1(t, 0) \sim \sqrt{2\pi}$  as  $t \rightarrow \infty$ , hence for  $p(x)$  defined in (2.1), we can approximate  $X$ 's moment generating function  $\Phi(t)$

$$\Phi(t) = \int_0^\infty e^{tx} p(x) dx = c\sqrt{2\pi}\sigma e^{K(\hat{x}, t)} (1 + o(1)). \tag{7.40}$$

If  $\alpha = 1$ , it holds as  $t \rightarrow \infty$ ,

$$T_1(t, 1) = -\frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^4 \exp \left( -\frac{y^2}{2} \right) dy = -\frac{\sqrt{2\pi}h''(\hat{x})\sigma^3}{2} (1 + o(1)),$$

hence we have with  $\Psi(t, \alpha)$  defined in Lemma 7.5

$$\Psi(t, 1) = -c\sqrt{2\pi}\sigma^2 e^{K(\hat{x}, t)} \frac{h''(\hat{x})\sigma^3}{2} (1 + o(1)) = -\Phi(t) \frac{h''(\hat{x})\sigma^4}{2} (1 + o(1)), \tag{7.41}$$

which, together with the definition of  $\Psi(t, \alpha)$ , yields

$$\int_0^\infty x e^{tx} p(x) dx = \Psi(t, 1) + \hat{x}\Phi(t) = \left( \hat{x} - \frac{h''(\hat{x})\sigma^4}{2} (1 + o(1)) \right) \Phi(t). \tag{7.42}$$

Hence we get

$$m(t) = \frac{d \log \Phi(t)}{dt} = \frac{\int_0^\infty x e^{tx} p(x) dx}{\Phi(t)} = \hat{x} - \frac{h''(\hat{x})\sigma^4}{2} (1 + o(1)). \tag{7.43}$$

Set  $\alpha = 2$ , as  $t \rightarrow \infty$ , it follows

$$\begin{aligned}\Psi(t, 2) &= c\sigma^3 e^{K(\hat{x}, t)} \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^2 \exp\left(-\frac{y^2}{2}\right) dy (1 + o(1)) \\ &= c\sqrt{2\pi}\sigma^3 e^{K(\hat{x}, t)} (1 + o(1)) = \sigma^2 \Phi(t) (1 + o(1)).\end{aligned}\quad (7.44)$$

Using (7.41), (7.43) and (7.44), we have

$$\begin{aligned}\int_0^\infty (x - m(t))^2 e^{tx} p(x) dx &= \int_0^\infty (x - \hat{x} + \hat{x} - m(t))^2 e^{tx} p(x) dx \\ &= \int_0^\infty (x - \hat{x})^2 e^{tx} p(x) dx + 2(\hat{x} - m(t)) \int_0^\infty (x - \hat{x}) e^{tx} p(x) dx + (\hat{x} - m(t))^2 \Phi(t) \\ &= \Psi(t, 2) + 2(\hat{x} - m(t)) \Psi(t, 1) + (\hat{x} - m(t))^2 \Phi(t) \\ &= \sigma^2 \Phi(t) (1 + o(1)) - h''(\hat{x}) \sigma^4 \left( \Phi(t) \frac{h''(\hat{x}) \sigma^4}{2} \right) (1 + o(1)) + \left( \frac{h''(\hat{x}) \sigma^4}{2} \right)^2 \Phi(t) (1 + o(1)) \\ &= \left( \sigma^2 - \frac{(h''(\hat{x}) \sigma^4)^2}{4} \right) \Phi(t) (1 + o(1)),\end{aligned}$$

thus we have

$$s^2(t) = \frac{d^2 \log \Phi(t)}{dt^2} = \frac{\int_0^\infty (x - m(t))^2 e^{tx} p(x) dx}{\Phi(t)} = \left( \sigma^2 - \frac{(h''(\hat{x}) \sigma^4)^2}{4} \right) (1 + o(1)). \quad (7.45)$$

Set  $\alpha = 3$ , the first term of  $T_1(t, 3)$  vanishes, we obtain as  $t \rightarrow \infty$

$$\begin{aligned}\Psi(t, 3) &= -c\sqrt{2\pi}\sigma^4 e^{K(\hat{x}, t)} \frac{h''(\hat{x})\sigma^3}{2} \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} \frac{1}{\sqrt{2\pi}} y^6 \exp\left(-\frac{y^2}{2}\right) dy \\ &= -cM_6 \sqrt{2\pi} e^{K(\hat{x}, t)} \frac{h''(\hat{x})\sigma^7}{2} (1 + o(1)) = -M_6 \frac{h''(\hat{x})\sigma^6}{2} \Phi(t) (1 + o(1)),\end{aligned}\quad (7.46)$$

where  $M_6$  denotes the sixth order moment of standard normal distribution. Using (7.41), (7.43),

(7.44) and (7.46), we have as  $t \rightarrow \infty$

$$\begin{aligned}
\int_0^\infty (x - m(t))^3 e^{tx} p(x) dx &= \int_0^\infty (x - \hat{x} + \hat{x} - m(t))^3 e^{tx} p(x) dx \\
&= \int_0^\infty \left( (x - \hat{x})^3 + 3(x - \hat{x})^2(\hat{x} - m(t)) + 3(x - \hat{x})(\hat{x} - m(t))^2 + (\hat{x} - m(t))^3 \right) e^{tx} p(x) dx \\
&= \Psi(t, 3) + 3(\hat{x} - m(t))\Psi(t, 2) + 3(\hat{x} - m(t))^2\Psi(t, 1) + (\hat{x} - m(t))^3\Phi(t) \\
&= -M_6 \frac{h''(\hat{x})\sigma^6}{2} \Phi(t)(1 + o(1)) + (3/2)h''(\hat{x})\sigma^4(\sigma^2\Phi(t))(1 + o(1)) \\
&\quad - 3\left(\frac{h''(\hat{x})\sigma^4}{2}\right)^2 \Phi(t) \frac{h''(\hat{x})\sigma^4}{2} (1 + o(1)) + \left(\frac{h''(\hat{x})\sigma^4}{2}\right)^3 \Phi(t)(1 + o(1)) \\
&= \left(\frac{3 - M_6}{2} h''(\hat{x})\sigma^6 - \frac{(h''(\hat{x})\sigma^4)^3}{4}\right) \Phi(t)(1 + o(1)),
\end{aligned}$$

hence we get

$$\mu_3(t) = \frac{d^3 \log \Phi(t)}{dt^3} = \frac{\int_0^\infty (x - m(t))^3 e^{tx} p(x) dx}{\Phi(t)} = \left(\frac{3 - M_6}{2} h''(\hat{x})\sigma^6 - \frac{(h''(\hat{x})\sigma^4)^3}{4}\right) (1 + o(1)). \quad (7.47)$$

Finally, we finish the proof by simplifying (7.43) (7.45) and (7.47).

**Case 1:**  $h \in R_\beta$ . We have gotten in (7.24)

$$h''(\hat{x}) = (\beta(\beta - 1) + o(1))\psi(t)^{\beta-2}l_0(\psi(t)),$$

where  $l_0 \in R_0$ . In (7.12), we have  $\sigma^2 \sim \psi(t)/(\beta t)$ , hence it holds

$$h''(\hat{x})\sigma^4 = \frac{\beta - 1}{\beta} \frac{\psi(t)^\beta}{t^2} l_0(\psi(t))(1 + o(1)) = \frac{\beta - 1}{\beta} \frac{l_0(\psi(t))l_1(t)^\beta}{t} (1 + o(1)),$$

where last equality holds since  $\psi(t) \sim t^{1/\beta}l_1(t)$  for some slowly varying function  $l_1$ . Obviously,  $h''(\hat{x})\sigma^4 = o(\hat{x})$ , thus we have

$$m(t) \sim \hat{x} = \psi(t).$$

It holds also as  $t \rightarrow \infty$

$$\frac{(h''(\hat{x})\sigma^4)^2}{\sigma^2} = (\beta - 1)^2 \frac{l_0(\psi(t))^2}{\psi(t)^2} (1 + o(1)) \rightarrow 0,$$

which implies  $(h''(\hat{x})\sigma^4)^2 = o(\sigma^2)$ . Therefore it follows

$$s^2(t) \sim \sigma^2 = \psi'(t). \quad (7.48)$$

For  $\mu_3$ , it holds  $(h''(\hat{x})\sigma^4)^3 = o(h''(\hat{x})\sigma^6)$  since

$$\frac{(h''(\hat{x})\sigma^4)^3}{h''(\hat{x})\sigma^6} = h''(\hat{x})^2\sigma^6 = \frac{(\beta-1)^2}{\beta} \frac{\psi(t)^{2\beta-1}l_0(\psi(t))^2}{t^3}(1+o(1)) \longrightarrow 0,$$

where last step holds from the fact  $\psi(t)^{2\beta-1}/t^3 \sim l_1(t)^{2\beta-1}/t^{1+1/\beta}$ . We have

$$\mu_3(t) \sim \frac{3-M_6}{2}h''(\hat{x})\sigma^6. \quad (7.49)$$

It is straightforward that (7.14) holds for  $h \in R_\beta$ , thus  $h''(\hat{x})\sigma^6 = -\psi''(t)/(\psi'(t))^3 * (\psi'(t))^3 = -\psi''(t)$ . We get

$$\mu_3(t) \sim \frac{M_6-3}{2}\psi''(t).$$

**Case 2:** If  $h \in R_\infty$ , recall that we have obtained in (7.26)

$$h''(\hat{x}) = -\frac{\psi''(t)}{(\psi'(t))^3} = \frac{t}{\psi^2(t)\epsilon^2(t)}(1+o(1)),$$

consider  $\sigma^2 = \psi'(t) = \psi(t)\epsilon(t)/t$ , it holds

$$h''(\hat{x})\sigma^4 = \frac{1}{t}(1+o(1)).$$

Notice  $h''(\hat{x})\sigma^4 = o(\hat{x})$  as  $t \rightarrow \infty$ , hence it holds

$$m(t) \sim \hat{x} = \psi(t).$$

And as  $t \rightarrow \infty$  it holds  $(h''(\hat{x})\sigma^4)^2 \sim 1/t^2 = o(\sigma^2)$ , thus we obtain

$$s^2(t) \sim \sigma^2 = \psi'(t).$$

As regards to  $\mu_3(t)$ , we have  $(h''(\hat{x})\sigma^4)^3 \sim 1/t^3$ , but  $h''(\hat{x})\sigma^6 \sim \psi(t)\epsilon(t)/t^2$ , hence it holds  $(h''(\hat{x})\sigma^4)^3 = o(h''(\hat{x})\sigma^6)$ . It follows

$$\mu_3(t) \sim \frac{M_6-3}{2}\psi''(t).$$

**Proof of Corollary 3.1 : Case 1:** If  $h \in R_\beta$ . By (7.48) and (7.49), it holds as  $t \rightarrow \infty$

$$\frac{\mu_3}{s^3} \sim \frac{M_6-3}{2}h''(\hat{x})\sigma^3. \quad (7.50)$$

Then using (7.24) and (7.12), we get for  $l_0 \in R_0$

$$\begin{aligned} h''(\hat{x})\sigma^3 &\sim \beta(\beta-1)\psi(t)^{\beta-2}l_0(\psi(t))\left(\frac{\psi(t)}{\beta t}\right)^{3/2} \\ &= \frac{\beta-1}{\sqrt{\beta}}l_0(\psi(t))\frac{\psi(t)^{\beta-1/2}}{t^{3/2}} \longrightarrow 0, \end{aligned} \quad (7.51)$$

where last step holds since  $\psi(t) \sim t^{1/\beta}l_1(t)$  for some slowly varying function  $l_1(t)$ . (7.50) and (7.51) yields (3.1).

**Case 2:** If  $h \in R_\infty$ . In (1) we have showed it holds

$$\frac{\mu_3(t)}{s^3(t)} \sim \frac{M_6 - 3}{2} \frac{\psi''(t)}{\psi'(t)^{3/2}}.$$

By (7.16) and (7.17), we have as  $t \rightarrow \infty$

$$\frac{\psi''(t)}{\psi'(t)^{3/2}} \sim -\frac{\psi(t)\epsilon(t)}{t^2} \left(\frac{\psi(t)\epsilon(t)}{t}\right)^{-3/2} = -\frac{1}{\sqrt{t\psi(t)\epsilon(t)}} \longrightarrow 0,$$

where last step holds under condition (2.7). Hence the proof.

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